

# Refined Schur Method for Robust Pole Assignment with Repeated Poles

Zhen-Chen Guo, Jiang Qian, Yun-feng Cai and Shu-fang Xu

**Abstract**—Schur-type methods in [6] and [11] solve the robust pole assignment problem by employing the departure from normality of the closed-loop system matrix as the measure of robustness. They work well generally when all poles to be assigned are simple. However, when some poles are close or even repeated, the eigenvalues of the computed closed-loop system matrix might be inaccurate. In this paper, we present a refined Schur method, which is able to deal with the case when some or all of the poles to be assigned are repeated. More importantly, the refined Schur method can still be applied when `place` [14] and `robpole` [28] fail to output a solution when the multiplicity of some repeated poles is greater than the input freedom.

**Index Terms**—robust pole assignment, repeated poles, departure from normality.

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## I. INTRODUCTION

THE behavior of the state feedback control system in engineering is essentially determined by the eigen-structure of the closed-loop system matrix. Such observation ultimately evokes the arising of the pole assignment problem, which can be mathematically stated as follows. Denote the dynamic state equation of the time invariant linear system by

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where  $A \in \mathbb{R}^{n \times n}$  is the open-loop system matrix and  $B \in \mathbb{R}^{n \times m}$  is the input matrix. In control theory, the **State-Feedback Pole Assignment Problem (SFPA)** is to find a state feedback matrix  $F \in \mathbb{R}^{m \times n}$  such that the eigenvalues of the closed-loop system matrix  $A_c = A + BF$ , associated with the closed-loop system

$$\dot{x}(t) = Ax(t) + Bu(t) = (A + BF)x(t) = A_c x(t),$$

are the given poles in  $\mathcal{L} = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , which is closed under complex conjugate. Many valuable contributions have been made to the **SFPA**. We refer readers to [3], [8], [12], [17]–[20], [23], [29], [32] for details. It is well known that the **SFPA** is solvable for any  $\mathcal{L}$  if and only if  $(A, B)$  is controllable [31], [32]. Through the rest of this paper, we will always assume that  $(A, B)$  is controllable.

When  $m > 1$ , the solution to the **SFPA** is generally not unique. It then leads to the problem on how to explore the freedom of  $F$  such that the closed-loop system achieves some desirable properties. An important engineering application is to find an appropriate solution  $F \in \mathbb{R}^{m \times n}$  to the **SFPA** such that the eigenvalues of the closed-loop system matrix  $A_c = A + BF$  are as insensitive to perturbations on  $A_c$  as possible, which is known as the **State-Feedback Robust Pole Assignment Problem (SFRPA)**.

To solve the **SFRPA**, it is imperative to choose an appropriate measure of robustness to characterize the “insensitivity” quantitatively. Based on different measures, various methods [4]–[7], [9]–[11], [13]–[16], [21], [22], [24]–[26], [28], [30], [32] are put forward. The most attractive methods might be those given by Kautsky, Nichols, and

Van Dooren [14], where the adopted measures are closely related to the condition number of the eigenvectors matrix of  $A_c$ . Method 1 in [14] is implemented as the function `place` in the MATLAB control system toolbox. Method 0 in [14] may not converge, and then Tits and Yang [28] posed a new approach upon it, which tends to maximize the absolute value of the determinant of the eigenvectors matrix of  $A_c$  and is implemented as the function `robpole` (from SLICOT). Based on recurrent neural networks, a method recently is put forward in [16], where many parameters need to be adjusted in order to achieve fast convergence. Notice that these methods can deal with both simple and repeated poles. However, they are iterative methods and hence can be expensive. Moreover, in these methods, the multiplicity of any repeated pole  $\lambda \in \mathcal{L}$  must not exceed the input freedom  $m$ . Otherwise, they will fail to give a solution. There exist feasible methods ([22], [24]) when the multiplicity of some repeated pole exceeds the input freedom  $m$ . They also tend to minimize the condition number of the eigenvectors matrix of  $A_c$ . In both methods, the real Jordan canonical form of the closed-loop system matrix is employed, and the size of each Jordan block of the repeated poles is assumed to be known in prior, which is, however, generally hard to obtain. Additionally, both methods could be numerical unstable since the computation of the Jordan canonical form of a matrix is usually suspected.

Another type of methods uses the departure from normality of  $A_c$  as the measure of robustness. It is firstly proposed as the **SCHUR** method in [6]. Some variations can also be found there. Recently, the authors [11] made some improvements to the methods proposed in [6], especially for placing complex conjugate poles, which is referred to as the **Schur-rob** method. All these Schur-type methods are designed for the case when all poles to be assigned are simple. If some poles are close or even repeated, these methods can still output a solution  $F$ , but the relative errors of the eigenvalues of the computed closed-loop system matrix  $A_c = A + BF$ , compared with the entries in  $\mathcal{L}$ , might be fairly large.

In this paper, we intend to propose a refined version of the **Schur-rob** method [11] specifically for repeated poles. It is well known that a defective eigenvalue, whose geometric multiplicity is less than its algebraic multiplicity, is generally more sensitive to perturbations than a semi-simple one, whose geometric and algebraic multiplicities are identical. So in the present refined Schur method, we manage to keep the geometric multiplicities of the repeated poles as large as possible by constructing the real Schur form of  $A_c$  in more special form, and then attempt to minimize the departure from normality of  $A_c$ . The present refined Schur method can achieve higher relative accuracy of the placed poles than those Schur-type methods in [6], [11] for repeated poles. Moreover, it still works well when methods in [14], [28] fail in the case where the multiplicity of some poles is greater than  $m$ . Numerical examples illustrate the superiorities of our approach.

The rest of this paper is organized as follows. Section II displays some useful preliminaries for solving the **SFRPA**. Our refined Schur method to assign repeated poles is developed in Section III. Several illustrative examples are presented in Section IV to illustrate the performance of our method. Some concluding remarks are finally

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drawn in Section V.

## II. PRELIMINARIES AND NOTATIONS

We first briefly review the parametric solutions to the **SFPA** [6], [11] using the real Schur decomposition of the closed-loop system matrix  $A_c = A + BF$ . Let

$$A + BF = XTX^\top \quad (1)$$

be the real Schur decomposition of  $A_c$ , where  $X \in \mathbb{R}^{n \times n}$  is orthogonal and  $T \in \mathbb{R}^{n \times n}$  is upper quasi-triangular. Without loss of generality, assume that  $B$  is of full column rank and let  $B = Q[R^\top \ 0]^\top = [Q_1 \ Q_2][R^\top \ 0]^\top = Q_1 R$  be the QR decomposition of  $B$ , where  $Q \in \mathbb{R}^{n \times n}$  is orthogonal,  $R \in \mathbb{R}^{m \times m}$  is nonsingular upper triangular, and  $Q_1 \in \mathbb{R}^{n \times m}$ . Then with  $X$  and  $T$  satisfying

$$Q_2^\top (AX - XT) = 0, \quad (2)$$

the parametric solutions to the **SFPA** can be given by

$$F = R^{-1}Q_1^\top (XTX^\top - A).$$

Consequently, once the orthogonal  $X$  and the upper quasi-triangular  $T$  satisfying (2) are obtained,  $F$  will be acquired immediately.

When solving the **SFRPA**, we employ the departure from normality of  $A_c$  as the measure of robustness, which can be specified as ([27])

$$\Delta_F(A_c) = \sqrt{\|A_c\|_F^2 - \sum_{j=1}^n |\lambda_j|^2},$$

where  $\lambda_j$ ,  $j = 1, \dots, n$ , are the poles to be placed. As in [11], we write  $T = D + N$ , where  $D$  and  $N$  are the block diagonal part and the strictly upper quasi-triangular part of  $T$ , respectively. Let the  $2 \times 2$  diagonal blocks in  $D$  be of the form  $\begin{bmatrix} \text{Re}(\lambda) & \delta \text{Im}(\lambda) \\ -\frac{1}{\delta} \text{Im}(\lambda) & \text{Re}(\lambda) \end{bmatrix}$  with  $\text{Im}(\lambda) \neq 0$ ,  $0 \neq \delta \in \mathbb{R}$ . Then  $\Delta_F(A_c)$  can be reformulated as

$$\Delta_F(A_c) = \sqrt{\|N\|_F^2 + \sum_{\text{Im}(\lambda) \neq 0} (\delta - \frac{1}{\delta})^2 \text{Im}(\lambda)^2}, \quad (3)$$

where the summation is over all  $2 \times 2$  diagonal blocks in  $D$ . Hence, if some poles to be assigned are non-real, it is not only the corresponding part in  $N$  that contributes to  $\Delta_F(A_c)$ , but also that in  $D$ . Our method displayed in the next section is designed to solve the **SFRPA** by finding some appropriate  $X$  and  $T$ , which satisfy (2), such that the departure from normality of  $A_c$ , specified in (3), is minimized. Acquiring an optimal solution to  $\min \Delta_F(A_c)$  is rather difficult. So instead of obtaining a global optimal solution, we prefer to get a suboptimal one with lower computational costs. The matrices  $X$  and  $T$  satisfying (2) are computed column by column via solving a series of optimization problems. Specifically, corresponding to a real pole  $\lambda_j$  (the  $j$ -th diagonal element in  $D$ ), the objective function to be minimized, associated with  $\Delta_F^2(A_c)$ , is  $\|v_j\|_2^2$ , where  $\check{v}_j = [v_j^\top \ 0]^\top$  with  $v_j \in \mathbb{R}^{j-1}$  is the  $j$ -th column of  $N$ ; while corresponding to a pair of complex conjugate poles  $\lambda_j, \lambda_{j+1} = \bar{\lambda}_j$ , it is

$$\|v_j\|_2^2 + \|v_{j+1}\|_2^2 + \text{Im}(\lambda_j)^2 (\delta - \frac{1}{\delta})^2, \quad (4)$$

where  $\check{v}_{j+k} = [v_{j+k}^\top \ 0]^\top$  with  $v_{j+k} \in \mathbb{R}^q$ ,  $q \leq j$ , are the  $(j+k)$ -th columns of  $N$  for  $k = 0, 1$ , and  $\begin{bmatrix} \text{Re}(\lambda_j) & \delta \text{Im}(\lambda_j) \\ -\frac{1}{\delta} \text{Im}(\lambda_j) & \text{Re}(\lambda_j) \end{bmatrix}$  is the corresponding  $2 \times 2$  diagonal block in  $D$ .

The following two lemmas are needed when assigning complex conjugate poles.

**Lemma II.1.** Let  $A, B \in \mathbb{R}^{n \times n}$  be symmetric, then there exist a diagonal matrix  $\Theta = \text{diag}(\theta_1, \theta_2, \dots, \theta_n)$  with  $\theta_j \geq 0$  ( $j = 1, 2, \dots, n$ ) and an orthogonal matrix  $U \in \mathbb{R}^{2n \times 2n}$ , whose  $j$ -th column  $u_j$  and  $(n+j)$ -th column  $u_{n+j}$  satisfy  $u_{n+j} = \begin{bmatrix} I_n & -I_n \end{bmatrix} u_j$ , such that

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} = U \text{diag}(\Theta, -\Theta) U^\top. \quad (5)$$

Furthermore, it holds that  $\begin{bmatrix} B & -A \\ -A & -B \end{bmatrix} = U \begin{bmatrix} 0 & -\Theta \\ -\Theta & 0 \end{bmatrix} U^\top$ .

Lemma II.1 can be verified directly by utilizing properties of Hamiltonian matrices, and we skip the proof here.

**Lemma II.2.** (Jacobi Orthogonal Transformation [11]) Assume that  $x, y \in \mathbb{R}^n$  are linearly independent, then there exists an orthogonal matrix  $Q \in \mathbb{R}^{2 \times 2}$ , such that  $\tilde{x}^\top \tilde{y} = 0$  with  $\begin{bmatrix} \tilde{x} & \tilde{y} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} Q$ .

Actually, the  $2 \times 2$  orthogonal matrix  $Q$  in Lemma II.2 can be obtained as follows. Let  $\varrho_1 = \|x\|_2^2$ ,  $\varrho_2 = \|y\|_2^2$ ,  $\gamma = x^\top y$ ,  $\tau = \frac{\varrho_2 - \varrho_1}{2\gamma}$  and define  $t$  as

$$t = \begin{cases} 1/(\tau + \sqrt{1 + \tau^2}), & \text{if } \tau \geq 0, \\ -1/(-\tau + \sqrt{1 + \tau^2}), & \text{if } \tau < 0. \end{cases}$$

Then the required  $Q$  is  $Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ , where  $c = 1/\sqrt{1 + t^2}$  and  $s = tc$ .

Throughout this paper, we denote the space spanned by the columns of a matrix  $M$  by  $\mathcal{R}(M)$ , the null space by  $\mathcal{N}(M)$ , and the set of eigenvalues of  $M$  by  $\lambda(M)$ . The MATLAB expression, which specifies the submatrix with the colon notation, will be used when necessary, that is,  $M(k : l, s : t)$  refers to the submatrix of  $M$  formed by rows  $k$  to  $l$  and columns  $s$  to  $t$ . We denote  $X = [x_1 \ x_2 \ \dots \ x_n]$  and  $X_j = [x_1 \ \dots \ x_j]$ . Write the strictly upper quasi-triangular part  $N$  of  $T$  as  $N = [\check{v}_1 \ \check{v}_2 \ \dots \ \check{v}_n]$ . For simplicity, we also denote  $T(1 : j, 1 : j)$  by  $T_j$ .

## III. REFINED SCHUR METHOD FOR REPEATED POLES

The method in [11] can dispose both simple and repeated poles. However, the repeated eigenvalues of the computed  $A_c$ , compared with the entries in  $\mathcal{L}$ , might be inaccurate. So this paper is specifically dedicated to repeated poles, both real and non-real. As pointed out in the Introduction part, a semi-simple eigenvalue is less sensitive to perturbations than a defective one. Thus when solving the **SFRPA**, we would keep the geometric multiplicities of repeated poles, as eigenvalues of  $A_c$ , as large as possible, which is actualized by setting special structure in the upper quasi-triangular matrix  $T$  in (1).

Analogously to [6], [11], we compute  $X$  and  $T$  satisfying (2) column by column, minimizing corresponding functions associated with  $\Delta_F^2(A_c)$  for real poles or complex conjugate poles. We start with the first pole  $\lambda_1$ , which is assumed to be repeated with multiplicity  $a_1 (> 1)$ , that is, it appears exactly  $a_1$  times in  $\mathcal{L}$ .

### A. Assigning repeated poles $\lambda_1$

The strategies vary depending on whether  $\lambda_1$  is real or non-real.

1)  $\lambda_1$  is real: As an eigenvalue of  $A_c = A + BF$ , denote its geometric multiplicity by  $g_1$ . It then follows that  $g_1 \leq m$  ([14]). If  $a_1 \leq m$ , the methods in [14], [28] can be applied, assigning  $\lambda_1$  as a semi-simple eigenvalue. Otherwise, that is  $a_1 > m$ , those methods will fail. In our refined Schur method, if  $a_1 \leq m$ ,  $\lambda_1$  can also be placed as a semi-simple eigenvalue of  $A_c$  with  $g_1 = a_1$ ; if  $a_1 > m$ ,  $\lambda_1$  can still be assigned with  $g_1 = m$ . Notice that geometric

multiplicity issues are not involved in those Schur-type methods in [6], [11].

Comparing the first  $a_1$  columns of (2) brings

$$Q_2^\top A X_{a_1} = Q_2^\top X_{a_1} T_{a_1}, \quad (6)$$

where  $X_{a_1} = X(:, 1 : a_1)$  satisfying  $X_{a_1}^\top X_{a_1} = I_{a_1}$  and  $T_{a_1} = T(1 : a_1, 1 : a_1)$  with  $\lambda(T_{a_1}) = \underbrace{\{\lambda_1, \dots, \lambda_1\}}_{a_1}$  are to be determined.

More specifically, to maximize the geometric multiplicity  $g_1$ , we take  $T_{a_1}$  in the special form of

$$T_{a_1} = \begin{bmatrix} n_1 & n_2 & \cdots & n_l \\ D_{11}(\lambda_1) & * & \cdots & * \\ & D_{22}(\lambda_1) & \cdots & * \\ & & \ddots & \vdots \\ & & & D_{ll}(\lambda_1) \end{bmatrix} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_l \end{matrix} \quad (7)$$

with  $D_{kk}(\lambda_1) = \lambda_1 I_{n_k}$ ,  $k = 1, \dots, l$ ,  $n_1 + \dots + n_l = a_1$ . The integers  $n_k$ ,  $k = 1, \dots, l$ , are also to be specified. Once such  $X_{a_1}$  and  $T_{a_1}$  satisfying (6) are found, the geometric multiplicity of  $\lambda_1$  will be no less than  $\max\{n_k : k = 1, \dots, l\}$ . So we shall make these  $n_k$  as large as possible. In the following, we show how to set these  $n_1, \dots, n_l$  and obtain the corresponding columns of  $X_{a_1}$  and  $T_{a_1}$  meanwhile.

Since  $D_{11}(\lambda_1) = \lambda_1 I_{n_1}$ , by equalling the first  $n_1$  columns in both sides of the equation in (6) and noticing the orthonormal requirements on columns of  $X$ , it shows that the first  $n_1$  columns of  $X$  should satisfy

$$\begin{aligned} M_1 [x_1 \ \cdots \ x_{n_1}] &= 0, \\ [x_1 \ \cdots \ x_{n_1}]^\top [x_1 \ \cdots \ x_{n_1}] &= I_{n_1}, \end{aligned} \quad (8)$$

where

$$M_1 = Q_2^\top (A - \lambda_1 I_n). \quad (9)$$

Here,  $M_1$  is of full row rank by the controllability of the matrix pencil  $(A, B)$ , which implies that  $\dim(\mathcal{N}(M_1)) = m$ . Let the columns of  $S \in \mathbb{R}^{n \times m}$  be an orthonormal basis of  $\mathcal{N}(M_1)$ . We then display how to determine  $n_1$  and find corresponding  $X_{n_1} = [x_1 \ \cdots \ x_{n_1}]$  by distinguishing two different situations.

a) *Situation I* —  $a_1 \leq m$  : In this situation, we set  $n_1 = a_1$ . Then by selecting  $x_1, x_2, \dots, x_{a_1} \in \mathcal{R}(S)$  with  $[x_1 \ x_2 \ \cdots \ x_{a_1}]^\top [x_1 \ x_2 \ \cdots \ x_{a_1}] = I_{a_1}$ , we have already assigned all  $\lambda_1$  and then proceed to the next pole as described in the next subsection — Subsection III-B. It is worthwhile to point out that with such choice, the geometric multiplicity  $g_1$  of  $\lambda_1$  is just  $a_1$ , that is,  $\lambda_1$  is a semi-simple eigenvalue of  $A_c$ .

b) *Situation II* —  $a_1 > m$  : In this situation, we can at most choose  $m$  orthonormal vectors from  $\mathcal{N}(M_1)$ . So we set  $n_1 = m$ , and then choose  $X_{n_1} = SZ$  with  $Z \in \mathbb{R}^{m \times m}$  being some orthogonal matrix.

Now assume that we have already obtained  $X_q = [x_1 \ \cdots \ x_q]$  and  $T_q = T(1 : q, 1 : q)$  with

$$T_q = \begin{bmatrix} n_1 & n_2 & \cdots & n_{k-1} \\ D_{11}(\lambda_1) & * & \cdots & * \\ & D_{22}(\lambda_1) & \cdots & * \\ & & \ddots & \vdots \\ & & & D_{k-1,k-1}(\lambda_1) \end{bmatrix} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_{k-1} \end{matrix},$$

where  $k > 1$ ,  $\sum_{j=1}^{k-1} n_j = q$ ,  $n_1 = m$  and  $D_{jj}(\lambda_1) = \lambda_1 I_{n_j}$ ,  $j = 1, \dots, k-1$ . We will show how to determine  $n_k$ , the corresponding columns of  $X$  and the corresponding strictly block upper triangular part  $T(1 : q, q+1 : q+n_k)$  in  $T$ .

From (6) and (7), the  $(q+1)$ -th,  $\dots$ ,  $(q+n_k)$ -th columns of  $X$  and  $N$  must satisfy

$$\begin{bmatrix} x_{q+j}^\top & v_{q+j}^\top \end{bmatrix}^\top \in \mathcal{N}(M_{q,q}), \quad (10)$$

where  $\check{v}_{q+j}$ , the  $(q+j)$ -th column of  $N$ , is  $\check{v}_{q+j} = [v_{q+j}^\top \ 0]^\top$  with  $v_{q+j} \in \mathbb{R}^q$  for  $j = 1, \dots, n_k$ , and

$$M_{q,q} = \begin{bmatrix} Q_2^\top (A - \lambda_1 I_n) & -Q_2^\top X_q \\ X_q^\top & 0 \end{bmatrix}. \quad (11)$$

Suppose that the columns of

$$S_{q,q} = \begin{bmatrix} S_{q,q}^{(1)} \\ S_{q,q}^{(2)} \end{bmatrix} \quad \text{with } S_{q,q}^{(1)} \in \mathbb{R}^{n \times m}, \ S_{q,q}^{(2)} \in \mathbb{R}^{q \times m}, \quad (12)$$

form an orthonormal basis of  $\mathcal{N}(M_{q,q})$ , where  $\dim(\mathcal{R}(S_{q,q})) = m$  is guaranteed by Theorem 1 in Subsection III-C. Let  $S_{q,q}^{(1)} = U_{q,q} \Sigma_{q,q} V_{q,q}^\top = U_{q,q} \begin{bmatrix} \Sigma_{q,q}^1 & 0 \\ 0 & 0 \end{bmatrix} V_{q,q}^\top$  be the Singular Value Decomposition (SVD) of  $S_{q,q}^{(1)}$  with  $\text{rank}(S_{q,q}^{(1)}) = r_q$  and  $\Sigma_{q,q}^1 = \text{diag}(\sigma_{1,q}, \dots, \sigma_{r_q,q})$ ,  $\sigma_{1,q} \geq \dots \geq \sigma_{r_q,q} > 0$ . Keep in mind that  $a_1 - q$  is the number of the pole  $\lambda_1$  to be assigned, and  $r_q$  is the rank of  $S_{q,q}^{(1)}$ , which is the maximum number of orthonormal vectors  $x_{q+j}$  satisfying (10). We then need to distinguish whether  $a_1 - q \leq r_q$  or not these two cases to discuss how to determine  $n_k$  and get those  $x_{q+j}, v_{q+j}, j = 1, \dots, n_k$ . Note that if  $r_q = 0$ , there does not exist nonzero vector  $x_{q+j}$  satisfying (10), and hence the method will terminate. Fortunately, Theorem 1 in Subsection III-C can assure that  $r_q$  is always nonzero.

- **Case i:**  $(a_1 - q) \leq r_q$ . In this case, we can set  $n_k = a_1 - q$ , that is, we can assign the remaining  $\lambda_1$  together. From (10), to minimize the departure from normality in (3), it is natural to solve the following optimization problem

$$\min \| [v_{q+1} \ v_{q+2} \ \cdots \ v_{a_1}] \|_F^2 \quad (13a)$$

$$\text{s.t.} \begin{cases} M_{q,q} \begin{bmatrix} x_{q+1} & x_{q+2} & \cdots & x_{a_1} \\ v_{q+1} & v_{q+2} & \cdots & v_{a_1} \end{bmatrix} = 0, \\ [x_{q+1} \ \cdots \ x_{a_1}]^\top [x_{q+1} \ \cdots \ x_{a_1}] = I_{a_1-q}, \end{cases} \quad (13b)$$

for  $x_{q+1}, \dots, x_{a_1}$  and  $v_{q+1}, \dots, v_{a_1}$ . By the definition of  $S_{q,q}$  we know that there exists  $Z \in \mathbb{R}^{m \times (a_1-q)}$  being of full column rank, such that

$$\begin{aligned} [x_{q+1} \ x_{q+2} \ \cdots \ x_{a_1}] &= S_{q,q}^{(1)} Z, \\ [v_{q+1} \ v_{q+2} \ \cdots \ v_{a_1}] &= S_{q,q}^{(2)} Z. \end{aligned} \quad (14)$$

Hence, the optimization problem (13) is equivalent to

$$\min_{Z^\top S_{q,q}^{(1)\top} S_{q,q}^{(1)} Z = I_{a_1-q}} \text{tr}(Z^\top S_{q,q}^{(2)\top} S_{q,q}^{(2)} Z). \quad (15)$$

Let  $\hat{Z} = V_{q,q}^\top Z$  with  $\hat{Z} = [\hat{Z}_1^\top \ \hat{Z}_2^\top]^\top$ ,  $\hat{Z}_1 \in \mathbb{R}^{r_q \times (a_1-q)}$ . Using  $S_{q,q}^{(1)\top} S_{q,q}^{(1)} + S_{q,q}^{(2)\top} S_{q,q}^{(2)} = I_m$ , then the problem (15) is equivalent to

$$\min_{\hat{Z}_1^\top \Sigma_{q,q}^1 \hat{Z}_1 = I_{a_1-q}} \text{tr}(\hat{Z}^\top \hat{Z}). \quad (16)$$

Write  $\tilde{Z}_1 = \Sigma_{q,q}^1 \hat{Z}_1$ , then (16) equals to

$$\min_{\tilde{Z}_1^\top \tilde{Z}_1 = I_{a_1-q}} \text{tr}(\tilde{Z}_1^\top (\Sigma_{q,q}^1)^{-2} \tilde{Z}_1), \quad (17)$$

with  $\hat{Z}_2 = 0$ . Obviously, the minimum value  $\sum_{j=1}^{a_1-q} \frac{1}{\sigma_{j,q}^2}$  of (17) is obtained when  $\tilde{Z}_1 = [e_1 \ \cdots \ e_{a_1-q}]$ , suggesting that (15) achieves its minimum when

$$Z = V_{q,q} [e_1 \ \cdots \ e_{a_1-q}] \text{diag}\left(\frac{1}{\sigma_{1,q}}, \dots, \frac{1}{\sigma_{a_1-q,q}}\right).$$

Once such  $Z$  is obtained,  $x_{q+1}, \dots, x_{a_1}$  and  $v_{q+1}, \dots, v_{a_1}$  can be computed by (14). We may then update  $X_q$  and  $T_q$  as

$$\begin{aligned} X_{a_1} &= [X_q \quad x_{q+1} \quad x_{q+2} \quad \cdots \quad x_{a_1}] \in \mathbb{R}^{n \times a_1}, \\ T_{a_1} &= \left[ \begin{array}{c|ccc} T_q & v_{q+1} & v_{q+2} & \cdots & v_{a_1} \\ \hline & & & & \lambda_1 I_{a_1-q} \end{array} \right] \in \mathbb{R}^{a_1 \times a_1}, \end{aligned} \quad (18)$$

and proceed with the next pole  $\lambda_2$ .

- **Case ii:**  $(a_1 - q) > r_q$ . In this case, we can choose at most  $r_q$  orthonormal  $x_{q+j}$ ,  $j \geq 1$ . So we set  $n_k = r_q$  and let

$$\begin{aligned} [x_{q+1} \quad \cdots \quad x_{q+r_q}] &= U_{q,q}(:, 1:r_q), \\ [v_{q+1} \quad \cdots \quad v_{q+r_q}] &= S_{q,q}^{(2)} V_{q,q}(:, 1:r_q) (\Sigma_{q,q}^1)^{-1}. \end{aligned}$$

It can be easily verified that such  $x_{q+j}, v_{q+j}$ ,  $j = 1, \dots, r_q$ , satisfy (10). It is worthwhile to point out that in this case we do not need to solve an optimization problem similar to (13) in **Case i**, because the value of the objective function now is a constant when the constraints are satisfied. We can then update  $X_q$  and  $T_q$  as

$$\begin{aligned} X_{q+n_k} &= X_{q+r_q} \\ &= [X_q \quad x_{q+1} \quad x_{q+2} \quad \cdots \quad x_{q+r_q}] \in \mathbb{R}^{n \times (q+r_q)}, \\ T_{q+n_k} &= T_{q+r_q} \\ &= \left[ \begin{array}{c|ccc} T_q & v_{q+1} & \cdots & v_{q+r_q} \\ \hline & & & \lambda_1 I_{r_q} \end{array} \right] \in \mathbb{R}^{(q+r_q) \times (q+r_q)}. \end{aligned} \quad (19)$$

In this case, some  $\lambda_1$  are still unassigned. We can then pursue a similar process either in **Case i** or **Case ii** until all  $\lambda_1$  are placed.

Eventually,  $T_{a_1}$  being of the form (7) would be acquired. And this procedure is summarized in Algorithm 1.

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**Algorithm 1** Assigning real  $\lambda_1$

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**Input:**

$A, Q_2, \lambda_1 \in \mathbb{R}$  and  $a_1$  (the multiplicity of  $\lambda_1$ ).

**Output:**

Orthogonal  $X_{a_1}$  and upper triangular  $T_{a_1}$ .

- 1: Find  $S \in \mathbb{R}^{n \times m}$ , whose columns are an orthonormal basis of  $\mathcal{N}(M_1)$  defined in (9).
  - 2: **if**  $a_1 \leq m$  **then**
  - 3: Set  $X_{a_1} = SZ$  with  $Z \in \mathbb{R}^{m \times a_1}$  satisfying  $Z^\top Z = I_{a_1}$  and  $T_{a_1} = \lambda_1 I_{a_1}$ .
  - 4: **else**
  - 5: Set  $X_{a_1}(:, 1:m) = S$ ,  $T_{a_1}(1:m, 1:m) = \lambda_1 I_m$ ,  $q = m$ ;
  - 6: **while**  $q < a_1$  **do**
  - 7: Find  $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$  with  $S_1 \in \mathbb{R}^{n \times m}$ ,  $S_2 \in \mathbb{R}^{q \times m}$ , whose columns are an orthonormal basis of  $\mathcal{N}(M_{q,q})$  in (11);
  - 8: **if**  $(a_1 - q) \leq \text{rank}(S_1)$  **then**
  - 9: Solve the optimization problem (13);
  - 10: Update  $X_{a_1}(:, 1:q)$  and  $T_{a_1}(1:q, 1:q)$  by (18), set  $q = a_1$ .
  - 11: **else**
  - 12: Update  $X_{a_1}(:, 1:q)$  and  $T_{a_1}(1:q, 1:q)$  by (19), set  $q = q + \text{rank}(S_1)$ .
  - 13: **end if**
  - 14: **end while**
  - 15: **end if**
- 

2)  $\lambda_1$  is non-real: Let  $\lambda_1 = \alpha_1 + i\beta_1$ , where  $\alpha_1, \beta_1 \in \mathbb{R}$  and  $\beta_1 \neq 0$ . As the eigenvalue of  $A_c$ , its algebraic multiplicity is denoted by  $a_1$ . Then  $\bar{\lambda}_1 = \alpha_1 - i\beta_1$  is also an eigenvalue of  $A_c$  with algebraic multiplicity  $a_1$ . We are to assign all  $a_1$  complex conjugate pairs  $\{\lambda_1, \bar{\lambda}_1\}$  in turn, where the complex conjugate poles  $\lambda_1$  and  $\bar{\lambda}_1$  are placed simultaneously.

Comparing the first  $2a_1$  columns of (2) and recalling that  $X$  is orthogonal, one can show that  $T_{2a_1}$  and  $X_{2a_1}$  must satisfy

$$Q_2^\top A X_{2a_1} - Q_2^\top X_{2a_1} T_{2a_1} = 0, \quad X_{2a_1}^\top X_{2a_1} = I_{2a_1}, \quad (20)$$

with  $\lambda(T_{2a_1}) = \{\underbrace{\lambda_1, \dots, \lambda_1}_{a_1}, \underbrace{\bar{\lambda}_1, \dots, \bar{\lambda}_1}_{a_1}\}$ . There is a classical

strategy in [11] to get  $T_{2a_1}$  and  $X_{2a_1}$  satisfying (20). Here, the substantial refinement on the strategy in [11] is taking the geometric multiplicities of  $\lambda_1$  and  $\bar{\lambda}_1$  into account. That is, we would choose  $T_{2a_1}$  in a more special form:

$$T_{2a_1} = \begin{bmatrix} 2n_1 & 2n_2 & \cdots & 2n_l \\ D_{11}(\lambda_1) & * & \cdots & * \\ & D_{22}(\lambda_1) & \cdots & * \\ & & \ddots & \vdots \\ & & & D_{ll}(\lambda_1) \end{bmatrix} \begin{matrix} 2n_1 \\ 2n_2 \\ \vdots \\ 2n_l \end{matrix}, \quad (21)$$

where  $D_{kk}(\lambda_1) = \text{diag}(D(\delta_{1,k}(\lambda_1)), \dots, D(\delta_{n_k,k}(\lambda_1)))$  with

$$\begin{aligned} D(\delta_{p,k}(\lambda_1)) &= \begin{bmatrix} \text{Re}(\lambda_1) & \delta_{p,k}(\lambda_1) \text{Im}(\lambda_1) \\ -\frac{1}{\delta_{p,k}(\lambda_1)} \text{Im}(\lambda_1) & \text{Re}(\lambda_1) \end{bmatrix}, \quad 0 \neq \delta_{p,k}(\lambda_1) \in \mathbb{R} \end{aligned} \quad (22)$$

for  $p = 1, \dots, n_k$ ,  $k = 1, \dots, l$ , and  $\sum_{k=1}^l n_k = a_1$ . With such special form of  $T_{2a_1}$ , the geometric multiplicity  $g_1$  of  $\lambda_1$  (and  $\bar{\lambda}_1$ ), as a repeated eigenvalue of  $A_c$ , is no less than  $\max\{n_k : k = 1, \dots, l\}$ .

Similarly to the case when  $\lambda_1$  is real, we then tend to choose  $\max\{n_k : k = 1, \dots, l\}$  as large as possible while computing  $T_{2a_1}$  and  $X_{2a_1}$  satisfying (20). However, the placing procedure for the case when  $\lambda_1$  is real can not be easily extended to this non-real case. The reason is that for the repeated and non-real poles, it is not only those columns in  $N$  that contribute to  $\Delta_F(A_c)$ , but also those  $\delta_{p,k}$  in the diagonal blocks  $D(\delta_{p,k}(\lambda_1))$  in  $D$ , which may differ in each  $2 \times 2$  blocks of  $D$ . Let us take the first  $2n_1$  columns of  $X$  and  $T$  as an illustration. Assume that  $n_1$  is known (Indeed,  $n_1$  is also a parameter to be determined. We will discuss how to set  $n_1$  later.), then to find the first  $2n_1$  columns of  $X$  and  $T$  simultaneously, we need to solve the following optimization problem originated from minimizing  $\Delta_F(A_c)$  defined in (3):

$$\min_{\delta_{1,1}(\lambda_1), \dots, \delta_{n_1,1}(\lambda_1)} \beta_1^2 \left( (\delta_{1,1}(\lambda_1) - \frac{1}{\delta_{1,1}(\lambda_1)})^2 + \cdots \right. \quad (23a)$$

$$\left. + (\delta_{n_1,1}(\lambda_1) - \frac{1}{\delta_{n_1,1}(\lambda_1)})^2 \right) \quad (23b)$$

$$\text{s.t.} \quad Q_2^\top (A X_{2n_1} - X_{2n_1} D_{11}(\lambda_1)) = 0, \quad (23c)$$

$$X_{2n_1}^\top X_{2n_1} = I_{2n_1}. \quad (23d)$$

The above optimization problem is fairly difficult to solve. The associate optimization problems corresponding to other  $D_{kk}(\lambda_1)$ ,  $k > 1$  are even more ticklish to solve. Be aware that in the case considered in the above part when  $\lambda_1$  is real, those  $\delta_{p,1}(\lambda_1)$  vanish, and we only need to find the columns of  $X$  and  $T$  satisfying the two constraints. Hence, rather than acquiring the columns of  $X$  and  $T$  corresponding to each  $D_{kk}(\lambda_1)$  straightway, we shall compute those associated with  $D(\delta_{p,k}(\lambda_1))$ ,  $p = 1, \dots, n_k$ ,  $k = 1, \dots, l$ , alternately. That is, in each step, we only compute two more columns



of  $X$  and  $T$  corresponding to  $D(\delta_{p,k}(\lambda_1))$ . Bear in mind that those  $n_1, \dots, n_l$  are also to be determined in the assigning process such that  $\max\{n_k : k = 1, \dots, l\}$  is as large as possible.

We start with the first two columns of  $X$  and  $T$ . Comparing the first two columns of (20), we have

$$Q_2^\top A \begin{bmatrix} x_1 & x_2 \end{bmatrix} = Q_2^\top \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -\frac{\alpha_1}{\delta_{1,1}(\lambda_1)}\beta_1 & \delta_{1,1}(\lambda_1)\beta_1 \\ \alpha_1 & \end{bmatrix}, \quad (24)$$

$$x_1^\top x_2 = 0, \quad \|x_1\|_2 = \|x_2\|_2 = 1. \quad (25)$$

Note that the corresponding strictly upper quasi-triangular part in  $T$  vanishes here, and the corresponding objective function (4) now becomes  $\beta_1^2(\delta_{1,1}(\lambda_1) - \frac{1}{\delta_{1,1}(\lambda_1)})^2$ . Apparently, it achieves its minimum value 0 at  $\delta_{1,1}(\lambda_1) = 1$ . We then show how to find  $x_1$  and  $x_2$  satisfying (24) and (25) with  $\delta_{1,1}(\lambda_1) = 1$ . Similarly as in [11], it is equivalent to find  $x_1$  and  $x_2$  such that

$$Q_2^\top (A - \lambda_1 I_n)(x_1 + ix_2) = 0 \quad (26)$$

with (25) holding.

It holds that  $\dim(\mathcal{N}(Q_2^\top (A - \lambda_1 I_n))) = m$  since  $(A, B)$  is controllable. Assume that the columns of  $S \in \mathbb{C}^{n \times m}$  form an orthonormal basis of  $\mathcal{N}(Q_2^\top (A - \lambda_1 I_n))$ . Define  $S_1 = \text{Re}(S)$ ,  $S_2 = \text{Im}(S)$ . Then (26) implies that  $x_1 + ix_2 = (S_1 + iS_2)(y_1 + iy_2)$  for some  $y_1, y_2 \in \mathbb{R}^m$ , or equivalently

$$x_1 = S_1 y_1 - S_2 y_2, \quad x_2 = S_1 y_2 + S_2 y_1. \quad (27)$$

If we can choose  $y_1$  and  $y_2$  to satisfy  $x_1^\top x_2 + x_2^\top x_1 = 0$  and  $x_1^\top x_1 - x_2^\top x_2 = 0$ , then the normalized  $x_1$  and  $x_2$  will satisfy (25) and (26). Direct calculations show that

$$\begin{aligned} x_1^\top x_2 + x_2^\top x_1 &= \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix} H_1 \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix}^\top, \\ x_1^\top x_1 - x_2^\top x_2 &= \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix} H_2 \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix}^\top, \end{aligned} \quad (28)$$

with

$$\begin{aligned} H_1 &= \begin{bmatrix} S_1^\top S_2 + S_2^\top S_1 & S_1^\top S_1 - S_2^\top S_2 \\ S_1^\top S_1 - S_2^\top S_2 & -(S_1^\top S_2 + S_2^\top S_1) \end{bmatrix}, \\ H_2 &= \begin{bmatrix} S_1^\top S_1 - S_2^\top S_2 & -(S_1^\top S_2 + S_2^\top S_1) \\ -(S_1^\top S_2 + S_2^\top S_1) & S_2^\top S_2 - S_1^\top S_1 \end{bmatrix}. \end{aligned}$$

Since  $S^* S = I_m$ , it can be easily verified that  $S_1^\top S_2 = S_2^\top S_1$  and  $S_1^\top S_1 + S_2^\top S_2 = I_m$ . If  $S_1^\top S_2 = 0$  and  $S_1^\top S_1 = \frac{1}{2}I_m$ , then  $x_1^\top x_2 = 0$  and  $\|x_1\|_2 = \|x_2\|_2$  for any  $y_1 \in \mathbb{R}^m$  and  $y_2 \in \mathbb{R}^m$  due to (28). In this case, we may arbitrarily choose  $y_1$  and  $y_2$  with  $\|y_1\|_2 = \|y_2\|_2 = 1$ , then  $x_1$  and  $x_2$  computed by (27) satisfy (25) and (26) as required. If  $S_1^\top S_2 \neq 0$  or  $S_1^\top S_1 \neq \frac{1}{2}I_m$ , then  $\text{rank}(H_1) \geq 1$ . Now by Lemma II.1, assume that

$$H_1 = U \text{diag}(\Theta, -\Theta) U^\top, \quad H_2 = U \begin{bmatrix} 0 & -\Theta \\ -\Theta & 0 \end{bmatrix} U^\top,$$

where  $U$  is orthogonal whose  $j$ -th column  $u_j$  and  $(m+j)$ -th column  $u_{m+j}$  satisfy  $u_{m+j} = \begin{bmatrix} -I_m \\ I_m \end{bmatrix} u_j$ ,  $j = 1, \dots, m$ , and  $\Theta = \text{diag}(\theta_1, \theta_2, \dots, \theta_m)$  with  $\theta_j \geq 0$ ,  $j = 1, \dots, m$  and  $\theta_1 > 0$ . Then with

$$\begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix}^\top = U \begin{bmatrix} \mu & 1 & 0 & \dots & 0 & -\mu & 1 & 0 & \dots & 0 \end{bmatrix}^\top, \quad (29)$$

where  $\mu = \sqrt{\theta_2/\theta_1}$ , one can show that  $x_1$  and  $x_2$  computed by (27) satisfy  $x_1^\top x_2 = 0$  and  $\|x_1\|_2 = \|x_2\|_2$ . Thus the normalized  $x_1$  and  $x_2$ , i.e.  $x_1 \triangleq x_1/\|x_1\|_2$ ,  $x_2 \triangleq x_2/\|x_2\|_2$ , are the vectors desired. Overall, we can obtain  $X_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$  and  $T_2 = D(\delta_{1,1}(\lambda_1)) = D_0(\lambda_1) \triangleq \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}$  in either case.

Now assume that the first  $2q$  ( $1 \leq q < a_1$ ) columns of  $X$  and  $T$  have already been obtained with

$$Q_2^\top A X_{2q} = Q_2^\top X_{2q} T_{2q}, \quad X_{2q}^\top X_{2q} = I_{2q}, \quad (30)$$

we are to find the subsequent  $(2q+1)$ -th and  $(2q+2)$ -th columns of  $X$  and  $T$ . Here  $T_{2q}$  is of the form similar as (21):

$$T_{2q} = \begin{bmatrix} 2n_1 & \dots & 2n_{k-1} & 2n_k \\ D_{11}(\lambda_1) & \dots & * & * \\ & \ddots & \vdots & \vdots \\ & & D_{k-1,k-1}(\lambda_1) & * \\ & & & D_{kk}(\lambda_1) \end{bmatrix} \begin{bmatrix} 2n_1 \\ \vdots \\ 2n_{k-1} \\ 2n_k \end{bmatrix}, \quad (31)$$

where  $D_{11}(\lambda_1), \dots, D_{kk}(\lambda_1)$  are block diagonal with  $2 \times 2$  matrices being of the form (22) as the diagonal blocks and  $n_1 + \dots + n_k = q$ . Notice that  $n_1, \dots, n_{k-1}$  have already been determined, while  $n_k$  might still be updated when computing the  $(2q+1)$ -th and  $(2q+2)$ -th columns of  $X$  and  $T$ . More specifically, denote

$$T_p = \begin{bmatrix} D_{11}(\lambda_1) & \dots & * \\ & \ddots & \vdots \\ & & D_{k-1,k-1}(\lambda_1) \end{bmatrix}$$

with  $p = 2n_1 + \dots + 2n_{k-1}$  and let  $D_{kk}(\lambda_1) = \text{diag}(D(\delta_{1,k}(\lambda_1)), \dots, D(\delta_{j,k}(\lambda_1)))$ , then the resulted  $T_{2q+2}$  could be in the form of

$$T_{2q+2} = \begin{bmatrix} T_p & * & v_{2q+1} & v_{2q+2} \\ & D_{kk}(\lambda_1) & 0 \\ & & D(\delta_{j+1,k}(\lambda_1)) \end{bmatrix}, \quad (32)$$

$v_{2q+1}, v_{2q+2} \in \mathbb{R}^p$ ,

or in the form of

$$T_{2q+2} = \left[ \begin{array}{c|cc} T_{2q} & v_{2q+1} & v_{2q+2} \\ \hline & D(\delta_{1,k+1}(\lambda_1)) & \end{array} \right], \quad v_{2q+1}, v_{2q+2} \in \mathbb{R}^{2q}. \quad (33)$$

If  $T_{2q+2}$  is in the form of (32),  $n_k$  will be increased by 1, meaning that  $n_k$  would be updated as  $n_k \triangleq n_k + 1$ ; while if  $T_{2q+2}$  is in the form of (33),  $n_k$  is fixed and  $n_{k+1}$  is initially set to be 1. Taking the geometric multiplicity  $g_1$  of  $\lambda_1$  (and  $\bar{\lambda}_1$ ) into account, we incline to make  $n_k$  as large as possible, suggesting that we would prefer  $T_{2q+2}$  in the form of (32) whenever possible.

We now turn to show how to determine whether (32) is possible and how to find the  $(2q+1)$ -th and  $(2q+2)$ -th columns of  $X$  and  $T$  accordingly. Provided that  $T_{2q+2}$  is in the form of (32), then by comparing the  $(2q+1)$ -th and  $(2q+2)$ -th columns of (20) and noting that  $X$  is orthogonal, we have

$$\begin{cases} Q_2^\top (A \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} - X_p \begin{bmatrix} v_{2q+1} & v_{2q+2} \end{bmatrix} - \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} D(\delta_{j+1,k}(\lambda_1))) = 0, \\ X_{2q}^\top \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} = 0, \\ \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix}^\top \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} = I_2. \end{cases} \quad (34)$$

Our goal now is to minimize (4) subject to (34). By writing  $\delta_{j+1,k}(\lambda_1) = \frac{\delta_2}{\delta_1}$  with  $0 \neq \delta_1 \in \mathbb{R}$  and  $\delta_2 \in \mathbb{R}$ , it follows from

[11] that the restriction (34) is equivalent to

$$\begin{cases} M_{2q,p} \begin{bmatrix} \tilde{x}_{2q+1} + i\tilde{x}_{2q+2} \\ \tilde{v}_{2q+1} + i\tilde{v}_{2q+2} \end{bmatrix} = 0, \\ \begin{bmatrix} \tilde{x}_{2q+1} & \tilde{x}_{2q+2} \end{bmatrix}^\top \begin{bmatrix} \tilde{x}_{2q+1} & \tilde{x}_{2q+2} \end{bmatrix} = \text{diag}(1/\delta_1^2, 1/\delta_2^2), \\ x_{2q+1} = \delta_1 \tilde{x}_{2q+1}, \quad x_{2q+2} = \delta_2 \tilde{x}_{2q+2}, \\ v_{2q+1} = \delta_1 \tilde{v}_{2q+1}, \quad v_{2q+2} = \delta_2 \tilde{v}_{2q+2}, \end{cases} \quad (35)$$

where

$$M_{2q,p} = \begin{bmatrix} Q_2^\top (A - \lambda_1 I_n) & -Q_2^\top X_p \\ X_{2q}^\top & 0 \end{bmatrix}. \quad (36)$$

Let the columns of

$$S_{2q,p} = \begin{bmatrix} S_{2q,p}^{(1)} \\ S_{2q,p}^{(2)} \end{bmatrix} \begin{matrix} n \\ p \end{matrix}$$

be an orthonormal basis of  $\mathcal{N}(M_{2q,p})$ . We shall distinguish three cases upon  $\dim(\mathcal{R}(S_{2q,p}^{(1)}))$  to reveal the assigning process, i.e., to compute  $x_{2q+1}, x_{2q+2}, v_{2q+1}$  and  $v_{2q+2}$  such that (4) is optimized.

- **Case iii:**  $\dim(\mathcal{R}(S_{2q,p}^{(1)})) \geq 2$ . Let  $S_{2q,p}^{(1)} = U_{2q,p} \Sigma_{2q,p} V_{2q,p}^*$  be the SVD of  $S_{2q,p}^{(1)}$  with  $\sigma_1, \sigma_2$  being the first two largest singular values of  $S_{2q,p}^{(1)}$  and let  $\tilde{x}_1 = \text{Re}(U_{2q,p} e_1)$ ,  $\tilde{y}_1 = \text{Im}(U_{2q,p} e_1)$ . If  $\tilde{x}_1^\top \tilde{y}_1 = 0$  and  $\|\tilde{x}_1\|_2 = \|\tilde{y}_1\|_2 = \frac{\sqrt{2}}{2}$ , we take

$$\begin{aligned} x_{2q+1} &= \sqrt{2} \tilde{x}_1, & v_{2q+1} &= \sqrt{2} \text{Re}(S_{2q,p}^{(2)} V_{2q,p} e_1 / \sigma_1), \\ x_{2q+2} &= \sqrt{2} \tilde{y}_1, & v_{2q+2} &= \sqrt{2} \text{Im}(S_{2q,p}^{(2)} V_{2q,p} e_1 / \sigma_1). \end{aligned}$$

With such choice, (34) is satisfied with  $\delta_{j+1,k}(\lambda_1) = 1$ , which results in the third term in the function defined in (4) vanishing and the first two terms achieving  $2 \frac{1-\sigma_2^2}{\sigma_1^2}$ , a value that is a comparable multiple (less than 2) of its minimum (Please refer to [11] for details.). Otherwise, that is  $\tilde{x}_1^\top \tilde{y}_1 \neq 0$  or  $\|\tilde{x}_1\|_2 \neq \|\tilde{y}_1\|_2$ , the suboptimal technique for assigning complex conjugate poles in [11] is applied. Specifically, denote  $\tilde{x}_2 = \text{Re}(U_{2q,p} e_2)$ ,  $\tilde{y}_2 = \text{Im}(U_{2q,p} e_2)$  and define  $\tilde{X}_{2q,p} = [\tilde{x}_1 \quad \tilde{x}_2]$ ,  $\tilde{Y}_{2q,p} = [\tilde{y}_1 \quad \tilde{y}_2]$ ,  $w_1 = S_{2q,p}^{(2)} V_{2q,p} e_1 / \sigma_1$ ,  $w_2 = S_{2q,p}^{(2)} V_{2q,p} e_2 / \sigma_2$ , then we set

$$\begin{aligned} x_{2q+1} &= [\tilde{X}_{2q,p} \quad -\tilde{Y}_{2q,p}] \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^\top, \\ x_{2q+2} &= [\tilde{Y}_{2q,p} \quad \tilde{X}_{2q,p}] \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^\top, \\ v_{2q+1} &= [\text{Re}(w_1) \quad \text{Re}(w_2) \quad -\text{Im}(w_1) \quad -\text{Im}(w_2)] \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^\top, \\ v_{2q+2} &= [\text{Im}(w_1) \quad \text{Im}(w_2) \quad \text{Re}(w_1) \quad \text{Re}(w_2)] \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^\top, \end{aligned}$$

where  $[\gamma_1 \quad \gamma_2 \quad \zeta_1 \quad \zeta_2]^\top \in \mathbb{R}^4$  is to be chosen such that the function defined in (4) is optimized in some sense. We refer readers to [11] for more details on this suboptimal technique. Overall, the resulted  $T_{2q+2}$  will be in the form of (32) in this case.

- **Case iv:**  $\dim(\mathcal{R}(S_{2q,p}^{(1)})) = 1$  and  $\text{Re}(u), \text{Im}(u)$  are linearly independent. Here  $u$  is the left singular vector of  $S_{2q,p}^{(1)}$  corresponding to its unique nonzero singular value  $\sigma_1$ . In this case, suppose that  $S_{2q,p}^{(1)} \in \mathbb{R}^{n \times r}$ , and let  $V_{2q,p} \in \mathbb{R}^{r \times r}$  be the right singular vectors matrix of  $S_{2q,p}^{(1)}$ . Define  $\mathcal{N}_1(M_{2q,p}) = \{[u^\top \quad w^\top]^\top : w = S_{2q,p}^{(2)} V_{2q,p} [\frac{1}{\sigma_1} \quad \eta_2 \quad \dots \quad \eta_r]^\top, \eta_2, \dots, \eta_r \in \mathbb{C}\}$ , then in the sense of nonzero scaling,  $\mathcal{N}_1(M_{2q,p})$  is the unique subset of  $\mathcal{N}(M_{2q,p})$  satisfying  $z \in \mathbb{C}^n$ ,  $w \in \mathbb{C}^p$ ,  $z \neq 0$  with  $[z^\top \quad w^\top]^\top \in \mathcal{N}(M_{2q,p})$ . Write  $u = \text{Re}(u) + i\text{Im}(u) \in \mathbb{C}^n$ ,  $w = \text{Re}(w) + i\text{Im}(w) \in \mathbb{C}^p$ , then we have that  $\text{Re}(u), \text{Im}(u)$ ,

$\text{Re}(w)$  and  $\text{Im}(w)$  satisfy

$$\begin{cases} Q_2^\top (A [\text{Re}(u) \quad \text{Im}(u)] - X_p [\text{Re}(w) \quad \text{Im}(w)] \\ \quad - [\text{Re}(u) \quad \text{Im}(u)] D_0(\lambda_1)) = 0, \\ X_{2q}^\top [\text{Re}(u) \quad \text{Im}(u)] = 0, \end{cases}$$

and  $\|w\|_2^2 = \frac{1-\sigma_1^2}{\sigma_1^2} + |\eta_2|^2 + \dots + |\eta_r|^2$ .

Since  $\text{Re}(u)$  and  $\text{Im}(u)$  are linearly independent, we shall pursue the Jacobi orthogonal transformation in Lemma II.2 on them, i.e.,  $[\tilde{x}_{2q+1} \quad \tilde{x}_{2q+2}] = [\text{Re}(u) \quad \text{Im}(u)] \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ , and set  $x_{2q+1}, x_{2q+2}$  be the normalized vectors of  $\tilde{x}_{2q+1}, \tilde{x}_{2q+2}$ , respectively. Accordingly,  $v_{2q+1}, v_{2q+2}$  are defined as

$$\begin{bmatrix} v_{2q+1} & v_{2q+2} \end{bmatrix} = [\text{Re}(w) \quad \text{Im}(w)] \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \frac{1}{\|\tilde{x}_{2q+1}\|_2} & \\ & \frac{1}{\|\tilde{x}_{2q+2}\|_2} \end{bmatrix}. \quad (37)$$

It is worthwhile to stress again that now we have  $\check{v}_{2q+s} = [v_{2q+s}^\top \quad 0]^\top$ ,  $v_{2q+s} \in \mathbb{R}^p$  for  $s = 1, 2$ . Be aware that  $w$  is unknown here since those values  $\eta_2, \dots, \eta_r \in \mathbb{C}$  have not been specified. Notice that  $D(\delta_{j+1,k}(\lambda_1))$  has already been determined with  $\delta_{j+1,k}(\lambda_1) = \frac{\|\tilde{x}_{2q+1}\|_2}{\|\tilde{x}_{2q+2}\|_2}$ , so we are to choose appropriate  $\eta_2, \dots, \eta_r$  to minimize  $\|v_{2q+1}\|_2^2 + \|v_{2q+2}\|_2^2$ , the first two terms of the function defined in (4).

Define  $S_{2q,p}^{(2)} V_{2q,p} = [w_1 \quad W]$  with  $w_1 \in \mathbb{C}^p$ ,  $Y_1 = [\text{Re}(W) \quad -\text{Im}(W)]$ ,  $Y_2 = [\text{Im}(W) \quad \text{Re}(W)]$ , and  $\text{Re}(y) + i\text{Im}(y) = y = [\eta_2 \quad \dots \quad \eta_r]^\top$ , then with some simple computations, we have

$$\begin{aligned} & \|v_{2q+1}\|_2^2 + \|v_{2q+2}\|_2^2 \\ &= [\text{Re}(y)^\top \quad \text{Im}(y)^\top] H [\text{Re}(y)^\top \quad \text{Im}(y)^\top]^\top \\ & \quad + g^\top [\text{Re}(y)^\top \quad \text{Im}(y)^\top]^\top + \zeta, \end{aligned} \quad (38)$$

where

$$\begin{aligned} H &= \frac{1}{\|\tilde{x}_{2q+1}\|_2^2} (cY_1 - sY_2)^\top (cY_1 - sY_2) \\ & \quad + \frac{1}{\|\tilde{x}_{2q+2}\|_2^2} (sY_1 + cY_2)^\top (sY_1 + cY_2), \\ g &= \frac{2}{\sigma_1} \left( \frac{c^2}{\|\tilde{x}_{2q+1}\|_2^2} + \frac{s^2}{\|\tilde{x}_{2q+2}\|_2^2} \right) Y_1^\top \text{Re}(w_1) \\ & \quad + \frac{2}{\sigma_1} \left( \frac{s^2}{\|\tilde{x}_{2q+1}\|_2^2} + \frac{c^2}{\|\tilde{x}_{2q+2}\|_2^2} \right) Y_2^\top \text{Im}(w_1) \\ & \quad + \frac{2cs}{\sigma_1} \left( \frac{1}{\|\tilde{x}_{2q+2}\|_2^2} - \frac{1}{\|\tilde{x}_{2q+1}\|_2^2} \right) (Y_2^\top \text{Re}(w_1) + Y_1^\top \text{Im}(w_1)), \\ \zeta &= \left( \frac{c^2}{\|\tilde{x}_{2q+1}\|_2^2} + \frac{s^2}{\|\tilde{x}_{2q+2}\|_2^2} \right) \frac{\|\text{Re}(w_1)\|_2^2}{\sigma_1^2} \\ & \quad + \left( \frac{s^2}{\|\tilde{x}_{2q+1}\|_2^2} + \frac{c^2}{\|\tilde{x}_{2q+2}\|_2^2} \right) \frac{\|\text{Im}(w_1)\|_2^2}{\sigma_1^2} \\ & \quad + \frac{2cs}{\sigma_1^2} \left( \frac{1}{\|\tilde{x}_{2q+2}\|_2^2} - \frac{1}{\|\tilde{x}_{2q+1}\|_2^2} \right) \text{Re}(w_1)^\top \text{Im}(w_1). \end{aligned}$$

Apparently,  $H$  is symmetric semipositive definite. We can further show that  $H$  is nonsingular, that is, it is positive definite. Indeed, assume that  $f \in \mathbb{R}^{2r-2}$  satisfies  $Hf = 0$ , which is then equivalent to  $Y_1 f = Y_2 f = 0$  by the definition of  $H$ . Using the definitions of  $Y_1, Y_2$  and  $W$ , we have

$$Y_1^\top Y_1 + Y_2^\top Y_2 = I_{2(r-1)}. \quad (39)$$

So it must hold that  $f = 0$ , which implies that  $H$  is symmetric positive definite. Consequently, the minimizer of (38) can be given by

$$[\text{Re}(y)^\top \quad \text{Im}(y)^\top]^\top = -\frac{1}{2} H^{-1} g.$$

Accordingly,  $v_{2q+1}$  and  $v_{2q+2}$  can be computed by (37). In all, in this case, the size of  $D_{kk}(\lambda_1)$  in  $T_{2q}$  is increased by 2, and  $T_{2q+2}$  being of the form of (32) will be obtained.

- **Case v:**  $\dim(\mathcal{R}(S_{2q,p}^{(1)})) = 1$  and  $\text{Re}(u), \text{Im}(u)$  are linearly dependent, or  $\dim(\mathcal{R}(S_{2q,p}^{(1)})) = 0$ . In this case, we cannot find  $x_{2q+1}, x_{2q+2}$  and  $v_{2q+1}, v_{2q+2} \in \mathbb{R}^p$  satisfying (34), meaning that  $T_{2q+2}$  cannot be chosen in the form of (32). Instead, we set  $T_{2q+2}$  in the form of (33) to continue the assigning process, which leads to:

$$\begin{cases} Q_2^\top (A \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} - X_{2q} \begin{bmatrix} v_{2q+1} & v_{2q+2} \end{bmatrix} \\ - \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} D(\delta_{1,k+1}(\lambda_1))) = 0, \\ X_{2q}^\top \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} = 0, \\ \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix}^\top \begin{bmatrix} x_{2q+1} & x_{2q+2} \end{bmatrix} = I_2, \end{cases} \quad (40)$$

with  $v_{2q+1}, v_{2q+2} \in \mathbb{R}^{2q}$ . Denote  $\delta_{1,k+1}(\lambda_1) = \frac{\delta_2}{\delta_1}$  with  $0 \neq \delta_1 \in \mathbb{R}$  and  $\delta_2 \in \mathbb{R}$ , then (40) is equivalent to some constraints similar to those in (35), where the essential difference here is that the parameter  $p$  in (35) is replaced by  $2q$ . More specifically, the matrix  $M_{2q,p}$  in (36) now turns to  $M_{2q,2q}$ , where the  $(1, 2)$  block is  $-Q_2^\top X_{2q}$  presently, instead of  $-Q_2^\top X_p$ . Bear in mind that now we have  $v_{2q+1} \in \mathbb{R}^{2q}$  and  $v_{2q+2} \in \mathbb{R}^{2q}$ , indicating that the  $2 \times 2$  block  $T(2q+1 : 2q+2, 2q+1 : 2q+2)$  locates in the  $(k+1)$ -th diagonal block  $D_{k+1,k+1}(\lambda_1)$  of  $T_{2a_1}$ . Now, we are to compute  $x_{2q+1}, x_{2q+2}$  and  $v_{2q+2}$  satisfying some nonlinear constraints such that the corresponding objective function specified as (4) is optimized.

The forthcoming Theorem 2 in Subsection III-C demonstrates that  $\dim(\mathcal{N}(M_{2q,2q})) = m$  and there exists  $[z^\top \ w^\top]^\top \in \mathcal{N}(M_{2q,2q})$  with  $z \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^{2q}$  such that  $z \neq 0$  and  $\text{Re}(z)$  and  $\text{Im}(z)$  are linearly independent, meaning that we can always find  $x_{2q+1}, x_{2q+2}, v_{2q+1}$  and  $v_{2q+2}$  to satisfy (40).

Suppose that the columns of  $S_{2q,2q} = [S_{2q,2q}^{(1)\top} \ S_{2q,2q}^{(2)\top}]^\top$  with  $S_{2q,2q}^{(1)} \in \mathbb{C}^{n \times m}$ ,  $S_{2q,2q}^{(2)} \in \mathbb{C}^{2q \times m}$ , form an orthonormal basis of  $\mathcal{N}(M_{2q,2q})$  and let  $S_{2q,2q}^{(1)} = U_{2q,2q} \Sigma_{2q,2q} V_{2q,2q}^*$  be the SVD of  $S_{2q,2q}^{(1)}$  with the singular values in decreasing order. Different placing strategies based on  $\text{rank}(S_{2q,2q}^{(1)})$  will be employed to acquire the  $(2q+1)$ -th and  $(2q+2)$ -th columns of  $X$  and  $T$ . Notice that Theorem 2 ensures that  $\text{rank}(S_{2q,2q}^{(1)}) \geq 1$ .

If  $\text{rank}(S_{2q,2q}^{(1)}) = 1$ , then  $S_{2q,2q}^{(1)}$  has only one nonzero singular value  $\sigma_1$  with  $u = U_{2q,2q} e_1$  being its corresponding left singular vector. Theorem 2 assures that  $\text{Re}(u)$  and  $\text{Im}(u)$  must be linearly independent. Then the assigning procedure is similar as that in **Case iv**. While  $\text{rank}(S_{2q,2q}^{(1)}) > 1$ , the assigning procedure is similar as that in **Case iii**.

Accordingly, in either situation, we can compute  $x_{2q+1}, x_{2q+2}, v_{2q+1}, v_{2q+2}$  with  $T_{2q+2}$  in the form of (33). Moreover, in this case,  $n_k$  is fixed, and  $n_{k+1}$  is initially set to be 1.

The above placing process can be proceeded with until all  $\{\lambda_1, \bar{\lambda}_1\}$  have been assigned. From the assigning process, we can see that if  $T_{2q} = D_{11}(\lambda_1)$  in (31),  $M_{2q,p}$  defined in (36) would be

$$M_{2q,0} = \begin{bmatrix} Q_2^\top (A - \lambda_1 I_n) \\ X_{2q}^\top \end{bmatrix},$$

where  $\text{rank}(M_{2q,0}) \leq (n-m)+2q$ . Thus provided that  $q \leq \lfloor \frac{m}{2} \rfloor - 1$ , we have  $\dim(\mathcal{N}(M_{2q,0})) \geq 2$ , which will lead the resulted  $(2q+2) \times (2q+2)$  leading principal submatrix  $T_{2q+2}$  of  $T$  in the form of (32), i.e.,  $T_{2q+2} = \text{diag}(T_{2q}, D(\delta_{q+1,1}(\lambda_1)))$ , suggesting that the size of the first diagonal block in  $T_{2a_1}$  is increased by 2. Consequently, in the case of  $a_1 \leq \lfloor \frac{m}{2} \rfloor$ , both  $\lambda_1$  and  $\bar{\lambda}_1$  can be placed with  $g_1 = a_1$ , that is, they are assigned as semi-simple eigenvalues of  $A_c = A + BF$ .

The procedure assigning  $\{\lambda_1, \bar{\lambda}_1\}$  is summarized in the following Algorithm 2.

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**Algorithm 2** Assigning complex conjugate  $\{\lambda_1, \bar{\lambda}_1\}$

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**Input:**

$A, Q_2, \lambda_1 \in \mathbb{C}$  with  $\text{Im}(\lambda_1) \neq 0$  and  $a_1$  (the multiplicity of  $\lambda_1$ ).

**Output:**

Orthogonal  $X_{2a_1}$  and upper quasi-triangular  $T_{2a_1}$ .

- 1: Find  $S = S_1 + iS_2$ , whose columns form an orthonormal basis of  $\mathcal{N}(Q_2^\top (A - \lambda_1 I_n))$ .
- 2: **if**  $S_1^\top S_2 = 0$  and  $S_1^\top S_1 = \frac{1}{2}I_m$  **then**
- 3:   Set  $y_1, y_2 \in \mathbb{R}^m$  be any vectors with  $\|y_1\|_2 = \|y_2\|_2 = 1$ ; compute  $x_1, x_2$  by (27) and set  $T_2 = D_0(\lambda_1)$ .
- 4: **else**
- 5:   Compute  $x_1, x_2$  by (27) with  $y_1, y_2 \in \mathbb{R}^m$  defined as in (29); normalize  $x_1, x_2$  and set  $T_2 = D_0(\lambda_1)$ .
- 6: **end if**
- 7: Set  $j = 2, k = 0$ .
- 8: **while**  $j < 2a_1$  **do**
- 9:   Find

$$S_{j,k} = \begin{bmatrix} S_{j,k}^{(1)} \\ S_{j,k}^{(2)} \end{bmatrix} \begin{matrix} n \\ k \end{matrix},$$

whose columns form an orthonormal basis of the null space

of  $M_{j,k} = \begin{bmatrix} Q_2^\top (A - \lambda_1 I_n) & -Q_2^\top X_k \\ X_j^\top & 0 \end{bmatrix}$ ; compute the SVD

of  $S_{j,k}^{(1)} = U_{j,k} \Sigma_{j,k} V_{j,k}^*$ .

- 10: **if**  $\text{rank}(S_{j,k}^{(1)}) \geq 2$  **then**
- 11:   Compute the  $(j+1)$ -th and  $(j+2)$ -th columns of  $X_{2a_1}$  and  $T_{2a_1}$  as in **Case iii**; set  $j = j + 2$ .
- 12: **else if**  $\text{rank}(S_{j,k}^{(1)}) = 1$  and  $\text{Re}(U_{j,k} e_1)$  and  $\text{Im}(U_{j,k} e_1)$  are linearly independent **then**
- 13:   Compute the  $(j+1)$ -th and  $(j+2)$ -th columns of  $X_{2a_1}$  and  $T_{2a_1}$  as in **Case iv**; set  $j = j + 2$ .
- 14: **else**
- 15:   Find

$$S_{j,j} = \begin{bmatrix} S_{j,j}^{(1)} \\ S_{j,j}^{(2)} \end{bmatrix} \begin{matrix} n \\ j \end{matrix},$$

whose columns form an orthonormal basis of the null space

of  $M_{j,j} = \begin{bmatrix} Q_2^\top (A - \lambda_1 I_n) & -Q_2^\top X_j \\ X_j^\top & 0 \end{bmatrix}$ ; compute the  $(j+1)$ -th and  $(j+2)$ -th columns of  $X_{2a_1}$  and  $T_{2a_1}$  as in **Case v**; set  $k = j$  and  $j = j + 2$ .

16: **end if**

17: **end while**

---

### B. Assigning repeated poles $\lambda_{j+1}$ ( $j \geq 1$ )

Suppose that the poles  $\lambda_1, \dots, \lambda_j$  have been assigned. Here the set  $\{\lambda_1, \dots, \lambda_j\}$  is closed under complex conjugate. That is, we have already obtained  $X_{r_0} = [x_1 \ x_2 \ \dots \ x_{r_0}] \in \mathbb{R}^{n \times r_0}$  and the  $r_0 \times r_0$  leading principal submatrix  $T_{r_0}$  of  $T$  satisfying

$$Q_2^\top (AX_{r_0} - X_{r_0}T_{r_0}) = 0, \quad X_{r_0}^\top X_{r_0} = I_{r_0},$$

where  $r_0 = \sum_{k=1}^j a_k$  with  $a_1, \dots, a_j$  being the multiplicities of  $\lambda_1, \dots, \lambda_j$ , respectively, and

$$\lambda(T_{r_0}) = \underbrace{\{\lambda_1, \dots, \lambda_1\}}_{a_1}, \dots, \underbrace{\{\lambda_j, \dots, \lambda_j\}}_{a_j} \subset \mathcal{L}.$$

Then we are to assign  $\lambda_{j+1}$  with multiplicity  $a_{j+1}$ . Here we assume  $a_{j+1} > 1$ . Similarly, we will again distinguish into two different cases when  $\lambda_{j+1}$  is real or non-real.

1)  $\lambda_{j+1}$  is real: To make the geometric multiplicity of  $\lambda_{j+1}$  as large as possible, we take  $T(r_0 + 1 : r_0 + a_{j+1}, r_0 + 1 : r_0 + a_{j+1})$ , the block diagonal part in  $T$  corresponding to  $\lambda_{j+1}$ , in the special form of

$$T(r_0 + 1 : r_0 + a_{j+1}, r_0 + 1 : r_0 + a_{j+1}) = \begin{bmatrix} n_1 & n_2 & \cdots & n_l \\ D_{11}(\lambda_{j+1}) & * & \cdots & * \\ & D_{22}(\lambda_{j+1}) & \cdots & * \\ & & \ddots & \vdots \\ & & & D_{ll}(\lambda_{j+1}) \end{bmatrix} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_l \end{matrix}, \quad (41)$$

where  $D_{kk}(\lambda_{j+1}) = \lambda_{j+1} I_{n_k}$ ,  $k = 1, \dots, l$ , and  $\sum_{k=1}^l n_k = a_{j+1}$ . With this form, the geometric multiplicity of  $\lambda_{j+1}$  will be no less than  $\max\{n_k : k = 1, \dots, l\}$ . Theoretically, if  $n_1 = a_{j+1}$ ,  $\lambda_{j+1}$  achieves its maximum geometric multiplicity and serves as a semi-simple eigenvalue of  $A_c$ , which is the most desirable. However,  $n_1$  can not be chosen to be equal to  $a_{j+1}$  in some cases.

The assigning process of obtaining the columns of  $X$  and  $T$  corresponding to the first diagonal block  $D_{11}(\lambda_{j+1})$  in (41) is as below. By noting the form of  $T(r_0 + 1 : r_0 + a_{j+1}, r_0 + 1 : r_0 + a_{j+1})$  in (41), then comparing the  $(r_0 + 1)$ -th to the  $(r_0 + n_1)$ -th columns of (2) shows that the corresponding columns of  $X$  and  $T$  must satisfy  $[x_{r_0+k}^\top \ v_{r_0+k}^\top]^\top \in \mathcal{N}(M_{r_0, r_0})$  for  $1 \leq k \leq n_1$ , where

$$M_{r_0, r_0} = \begin{bmatrix} Q_2^\top (A - \lambda_{j+1} I_n) & -Q_2^\top X_{r_0} \\ X_{r_0}^\top & 0 \end{bmatrix}, \quad (42)$$

and  $\check{v}_{r_0+k} = [v_{r_0+k}^\top \ 0]^\top$ ,  $v_{r_0+k} \in \mathbb{R}^{r_0}$  for  $1 \leq k \leq n_1$ . Let the columns of

$$S_{r_0, r_0} = \begin{bmatrix} S_{r_0, r_0}^{(1)} \\ S_{r_0, r_0}^{(2)} \end{bmatrix} \begin{matrix} n \\ r_0 \end{matrix} \quad (43)$$

be an orthonormal basis of  $\mathcal{N}(M_{r_0, r_0})$ . Write  $r_{r_0} = \text{rank}(S_{r_0, r_0}^{(1)})$ , which indicates that we can select at most  $r_{r_0}$  linearly independent vectors from  $\mathcal{R}(S_{r_0, r_0}^{(1)})$ . That is,  $n_1$  cannot exceed  $r_{r_0}$ . Similarly as the previous subsection — Subsection III-A1,  $r_{r_0}$  must be nonzero to assure that the assigning procedure would not interrupt. The related results are summarized in Theorem 1 in Subsection III-C. In the following, two different cases will be disposed separately.

- **Case i:**  $a_{j+1} \leq r_{r_0}$ . In this case, we set  $n_1 = a_{j+1}$ . With this choice,  $\lambda_{j+1}$  will act as a semi-simple eigenvalue of  $A_c$ . Then to get a small departure from normality of  $A_c$ , it is natural to consider the following optimization problem:

$$\min \| [v_{r_0+1} \ v_{r_0+2} \ \cdots \ v_{r_0+a_{j+1}}] \|_F^2 \quad (44a)$$

$$\text{s.t.} \begin{cases} M_{r_0, r_0} \begin{bmatrix} x_{r_0+1} & \cdots & x_{r_0+a_{j+1}} \\ v_{r_0+1} & \cdots & v_{r_0+a_{j+1}} \end{bmatrix} = 0, \\ \begin{bmatrix} x_{r_0+1} & \cdots & x_{r_0+a_{j+1}} \\ [x_{r_0+1} \ \cdots \ x_{r_0+a_{j+1}}]^\top \end{bmatrix} = I_{a_{j+1}}. \end{cases} \quad (44b)$$

Apparently, it can be solved by the same method that solves (13). Once the solution is obtained,  $X_{r_0}$  and  $T_{r_0}$  will be updated as

$$X_{r_0+a_{j+1}} = [X_{r_0} \ x_{r_0+1} \ \cdots \ x_{r_0+a_{j+1}}] \in \mathbb{R}^{n \times (r_0+a_{j+1})},$$

$$T_{r_0+a_{j+1}} = \begin{bmatrix} T_{r_0} & v_{r_0+1} & \cdots & v_{r_0+a_{j+1}} \\ \hline & \lambda_{j+1} I_{a_{j+1}} & & \end{bmatrix} \in \mathbb{R}^{(r_0+a_{j+1}) \times (r_0+a_{j+1})}, \quad (45)$$

where  $T_{r_0+a_{j+1}}$  is the  $(r_0 + a_{j+1}) \times (r_0 + a_{j+1})$  leading principal submatrix of  $T$ .

- **Case ii:**  $a_{j+1} > r_{r_0}$ . In this case, the maximum possible value of  $n_1$  is  $r_{r_0}$ , and we then set  $n_1 = r_{r_0}$ . Similarly to **Case ii** in Subsection III-A1, let  $S_{r_0, r_0}^{(1)} = U_{r_0, r_0} \Sigma_{r_0, r_0} V_{r_0, r_0}^\top$  be the SVD of  $S_{r_0, r_0}^{(1)}$  with  $\sigma_{1, r_0}, \dots, \sigma_{r_{r_0}, r_0}$  being its singular values, then we take

$$\begin{aligned} [x_{r_0+1} \ \cdots \ x_{r_0+r_{r_0}}] &= U_{r_0, r_0} [e_1 \ \cdots \ e_{r_{r_0}}], \\ [v_{r_0+1} \ \cdots \ v_{r_0+r_{r_0}}] &= S_{r_0, r_0}^{(2)} V_{r_0, r_0} [e_1 \ \cdots \ e_{r_{r_0}}] \text{diag}\left(\frac{1}{\sigma_{1, r_0}}, \dots, \frac{1}{\sigma_{r_{r_0}, r_0}}\right), \end{aligned}$$

and update  $X_{r_0}$  and  $T_{r_0}$  as

$$\begin{aligned} X_{r_0+n_1} &= X_{r_0+r_{r_0}} \\ &= [X_{r_0} \ x_{r_0+1} \ \cdots \ x_{r_0+r_{r_0}}] \in \mathbb{R}^{n \times (r_0+r_{r_0})}, \end{aligned}$$

$$\begin{aligned} T_{r_0+n_1} &= T_{r_0+r_{r_0}} \\ &= \begin{bmatrix} T_{r_0} & v_{r_0+1} & \cdots & v_{r_0+r_{r_0}} \\ \hline & \lambda_{j+1} I_{r_{r_0}} & & \end{bmatrix} \in \mathbb{R}^{(r_0+r_{r_0}) \times (r_0+r_{r_0})}. \end{aligned} \quad (46)$$

Hence, if  $a_{j+1} \leq r_{r_0}$ , all  $\lambda_{j+1}$  have been assigned, and we can continue with  $\lambda_{j+2}$ ; while in the case of  $a_{j+1} > r_{r_0}$ , we still need to perform a similar procedure as **Case i** and **Case ii** until all  $\lambda_{j+1}$  are assigned. Ultimately, we would acquire the  $(r_0 + a_{j+1}) \times (r_0 + a_{j+1})$  leading principal submatrix of  $T$  being of the form

$$\begin{aligned} T_{r_0+a_{j+1}} &= \begin{bmatrix} T_{r_0} & * & \cdots & * \\ & \lambda_{j+1} I_{n_1} & \cdots & * \\ & & \ddots & \vdots \\ & & & \lambda_{j+1} I_{n_l} \end{bmatrix} \in \mathbb{R}^{(r_0+a_{j+1}) \times (r_0+a_{j+1})}, \end{aligned}$$

where  $\sum_{k=1}^l n_k = a_{j+1}$ . Furthermore, the geometric multiplicity  $g_{j+1}$  of  $\lambda_{j+1}$  satisfies  $\max\{n_k : k = 1, \dots, l\} \leq g_{j+1} \leq m$ . We synthesize the assigning process of  $\lambda_{j+1}$  in Algorithm 3.

- 2)  $\lambda_{j+1}$  is non-real: Let  $\lambda_{j+1} = \alpha_{j+1} + i\beta_{j+1}$  with  $\alpha_{j+1}, \beta_{j+1} \in \mathbb{R}$  and  $\beta_{j+1} \neq 0$ . In this part, we shall sketch the process of assigning all complex conjugate pairs  $\{\lambda_{j+1}, \bar{\lambda}_{j+1}\}$ . Denote the algebraic multiplicity and geometric multiplicity of  $\lambda_{j+1}$  (and  $\bar{\lambda}_{j+1}$ ) by  $a_{j+1}$  and  $g_{j+1}$ , respectively. To make the geometric multiplicity  $g_{j+1}$  as large as possible, similarly as  $T_{2a_1}$  in Subsection III-A2, we take  $T(r_0 + 1 : r_0 + 2a_{j+1}, r_0 + 1 : r_0 + 2a_{j+1})$  in the special form of

$$\begin{aligned} T(r_0 + 1 : r_0 + 2a_{j+1}, r_0 + 1 : r_0 + 2a_{j+1}) &= \begin{bmatrix} 2n_1 & 2n_2 & \cdots & 2n_l \\ D_{11}(\lambda_{j+1}) & * & \cdots & * \\ & D_{22}(\lambda_{j+1}) & \cdots & * \\ & & \ddots & \vdots \\ & & & D_{ll}(\lambda_{j+1}) \end{bmatrix} \begin{matrix} 2n_1 \\ 2n_2 \\ \vdots \\ 2n_l \end{matrix}, \end{aligned} \quad (47)$$

where  $D_{kk}(\lambda_{j+1}) = \text{diag}(D(\delta_{1,k}(\lambda_{j+1})), \dots, D(\delta_{n_k,k}(\lambda_{j+1})))$  with

$$D(\delta_{p,k}(\lambda_{j+1})) = \begin{bmatrix} \text{Re}(\lambda_{j+1}) & \delta_{p,k}(\lambda_{j+1}) \text{Im}(\lambda_{j+1}) \\ -\frac{1}{\delta_{p,k}(\lambda_{j+1})} \text{Im}(\lambda_{j+1}) & \text{Re}(\lambda_{j+1}) \end{bmatrix}, \quad (48)$$



---

**Algorithm 3** Assigning real  $\lambda_{j+1}$ 


---

**Input:** $A, Q_2, X_{r_0}, T_{r_0}, \lambda_{j+1} \in \mathbb{R}$  and  $a_{j+1}$  (the multiplicity of  $\lambda_{j+1}$ ).**Output:**Orthogonal  $X_{r_0+a_{j+1}}$  and upper quasi-triangular  $T_{r_0+a_{j+1}}$ .1: Set  $q = 0$ .2: **while**  $q < a_{j+1}$  **do**

3: Find

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{matrix} n \\ r_0 + q \end{matrix},$$

whose columns form an orthonormal basis of  $\mathcal{N}(M_{r_0+q, r_0+q})$ , where

$$M_{r_0+q, r_0+q} = \begin{bmatrix} Q_2^\top (A - \lambda_{j+1} I_n) & -Q_2^\top X_{r_0+q} \\ X_{r_0+q}^\top & 0 \end{bmatrix};$$

4: **if**  $(a_{j+1} - q) \leq \text{rank}(S_1)$  **then**5: Solve the optimization problem (44) with  $r_0$  replaced by  $(r_0 + q)$  and  $a_{j+1}$  by  $(a_{j+1} - q)$ ;6: Update  $X_{r_0+q}$  and  $T_{r_0+q}$  similarly as (45), set  $q = a_{j+1}$ .7: **else**8: Update  $X_{r_0+q}$  and  $T_{r_0+q}$  similarly as (46), set  $q = q + \text{rank}(S_1)$ .9: **end if**10: **end while**


---

$0 \neq \delta_{p,k}(\lambda_{j+1}) \in \mathbb{R}$ ,  $p = 1, \dots, n_k$ ,  $k = 1, \dots, l$ , and  $\sum_{k=1}^l n_k = a_{j+1}$ . Apparently, as eigenvalues of  $A_c$ , the geometric multiplicity  $g_{j+1}$  of  $\lambda_{j+1}$  (and  $\bar{\lambda}_{j+1}$ ) is no less than  $\max\{n_k : k = 1, \dots, l\}$ .

Similarly as that in Subsection III-A2, we shall place one complex conjugate pair  $\{\lambda_{j+1}, \bar{\lambda}_{j+1}\}$  at a time, obtaining two columns of  $T$  and  $X$  corresponding to the  $2 \times 2$  matrix  $D(\delta_{p,k}(\lambda_{j+1}))$  concurrently.

Firstly, we dispose the issue that how to obtain the  $(r_0 + 1)$ -th and  $(r_0 + 2)$ -th columns of  $X$  and  $T$ . Notice that  $T(r_0 + 1 : r_0 + 2, r_0 + 1 : r_0 + 2) = D(\delta_{1,1}(\lambda_{j+1}))$ . Define  $\delta_{1,1}(\lambda_{j+1}) = \frac{\delta_2}{\delta_1}$  with  $0 \neq \delta_1 \in \mathbb{R}$  and  $\delta_2 \in \mathbb{R}$ , then it follows from [11] that

$$M_{r_0, r_0} \begin{bmatrix} \frac{1}{\delta_1} x_{r_0+1} + i \frac{1}{\delta_2} x_{r_0+2} \\ \frac{1}{\delta_1} v_{r_0+1} + i \frac{1}{\delta_2} v_{r_0+2} \end{bmatrix} = 0, \quad (49)$$

where the definition of  $M_{r_0, r_0}$  is analogous to that specified in (42) and  $\check{v}_{r_0+k} = [v_{r_0+k}^\top \ 0]^\top$ ,  $v_{r_0+k} \in \mathbb{R}^{r_0}$  for  $k = 1, 2$ . And the intrinsic changing on  $M_{r_0, r_0}$  is that now  $\lambda_{j+1} \in \mathbb{C}$  with  $\text{Im}(\lambda_{j+1}) \neq 0$ . Accordingly, to get proper  $x_{r_0+1}$ ,  $x_{r_0+2}$ ,  $v_{r_0+1}$ ,  $v_{r_0+2}$ ,  $\delta_1$  and  $\delta_2$ , we need to minimize the function defined in (4) subject to the two constraints (49) and  $[x_{r_0+1} \ x_{r_0+2}]^\top [x_{r_0+1} \ x_{r_0+2}] = I_2$ .

Theorem 2 in the forthcoming Subsection III-C shows that  $\dim(\mathcal{N}(M_{r_0, r_0})) = m$  and there exists  $[z^\top \ w^\top]^\top \in \mathcal{N}(M_{r_0, r_0})$  with  $0 \neq z \in \mathbb{C}^n$ ,  $w \in \mathbb{C}^{r_0}$  and  $\text{Re}(z)$ ,  $\text{Im}(z)$  being linearly independent. Define  $S_{r_0, r_0} = [S_{r_0, r_0}^{(1)\top} \ S_{r_0, r_0}^{(2)\top}]^\top$  with  $S_{r_0, r_0}^{(1)} \in \mathbb{C}^{n \times m}$ ,  $S_{r_0, r_0}^{(2)} \in \mathbb{C}^{r_0 \times m}$ , whose columns form an orthonormal basis of  $\mathcal{N}(M_{r_0, r_0})$ , the placing process will be realized through addressing two distinct cases upon  $\text{rank}(S_{r_0, r_0}^{(1)})$ . For convenience, we denote the left and right singular vectors of  $S_{r_0, r_0}^{(1)}$ , corresponding to its largest singular value  $\sigma_1$ , by  $u$  and  $v$ , respectively.

If  $\text{rank}(S_{r_0, r_0}^{(1)}) \geq 2$ , a similar placing process as that in **Case iii** in Subsection III-A2 will be implemented. That is, if  $\text{Re}(u)^\top \text{Im}(u) = 0$  and  $\|\text{Re}(u)\|_2 = \|\text{Im}(u)\|_2 = \frac{\sqrt{2}}{2}$ , we set  $x_{r_0+1} = \sqrt{2}\text{Re}(u)$ ,  $x_{r_0+2} = \sqrt{2}\text{Im}(u)$ , and  $v_{r_0+1} = \sqrt{2}\text{Re}(S_{r_0, r_0}^{(2)} v / \sigma_1)$ ,  $v_{r_0+2} = \sqrt{2}\text{Im}(S_{r_0, r_0}^{(2)} v / \sigma_1)$ . Otherwise, the complex conjugate pair placing strategy in [11] would be applied. When  $\text{rank}(S_{r_0, r_0}^{(1)}) = 1$ , Theorem 2 in the following subsection would guarantee that  $\text{Re}(u)$  and

$\text{Im}(u)$  are linearly independent. We then apply the Jacobi orthogonal transformation in Lemma II.2 to orthogonalize  $\text{Re}(u)$  and  $\text{Im}(u)$ , and then normalize the resulted vectors as  $x_{r_0+1}$  and  $x_{r_0+2}$ . Furthermore,  $v_{r_0+1}$  and  $v_{r_0+2}$  will be obtained by minimizing some function defined similarly as that in (38). The process resembles that in **Case iv** in Subsection III-A2, and we omit details here.

Now assume that we have obtained  $2q$  ( $1 \leq q < a_{j+1}$ ) columns of  $X$  and  $T$  corresponding to  $\{\lambda_{j+1}, \bar{\lambda}_{j+1}\}$ , we then proceed to compute the  $(r_0 + 2q + 1)$ -th and  $(r_0 + 2q + 2)$ -th columns of  $X$  and  $T$ , which virtually are associated with the diagonal block  $T(r_0 + 2q + 1 : r_0 + 2q + 2, r_0 + 2q + 1 : r_0 + 2q + 2)$  in  $T$ . The whole procedure is similar to what we do to get the  $(2q + 1)$ -th and  $(2q + 2)$ -th columns of  $X$  and  $T$  in Subsection III-A2, and we just give a concise presentation.

Assume that

$$T_{r_0+2q} = \left[ \begin{array}{c|ccc} T_{r_0} & * & \cdots & * \\ \hline & D_{11}(\lambda_{j+1}) & \cdots & * \\ & & \ddots & \vdots \\ & & & D_{tt}(\lambda_{j+1}) \end{array} \right],$$

where  $D_{kk}(\lambda_{j+1}) \in \mathbb{R}^{2n_k \times 2n_k}$ ,  $k = 1, \dots, t$ , and  $T(r_0 + 2q - 1 : r_0 + 2q, r_0 + 2q - 1 : r_0 + 2q) = D(\delta_{s,t}(\lambda_{j+1}))$ , indicating that  $T(r_0 + 2q - 1 : r_0 + 2q, r_0 + 2q - 1 : r_0 + 2q)$  is the  $s$ -th  $2 \times 2$  diagonal block in  $D_{tt}(\lambda_{j+1})$ . Denote  $p = r_0 + 2n_1 + \dots + 2n_{t-1}$ . Then like  $T_{2q+2}$  in Subsection III-A2,  $T_{r_0+2q+2}$  could be in the form of

$$T_{r_0+2q+2} = \left[ \begin{array}{c|cc} T_p & * & v_{r_0+2q+1} \ v_{r_0+2q+2} \\ \hline & D_{tt}(\lambda_{j+1}) & 0 \\ \hline & & D(\delta_{s+1,t}(\lambda_{j+1})) \end{array} \right], \quad (50)$$

$v_{r_0+2q+1}, v_{r_0+2q+2} \in \mathbb{R}^p$ ,

or

$$T_{r_0+2q+2} = \left[ \begin{array}{c|cc} T_{r_0+2q} & v_{r_0+2q+1} \ v_{r_0+2q+2} \\ \hline & D(\delta_{1,t+1}(\lambda_{j+1})) \end{array} \right],$$

$v_{r_0+2q+1}, v_{r_0+2q+2} \in \mathbb{R}^{r_0+2q}$ .

And to get a large  $g_{j+1}$ , we incline to  $T_{r_0+2q+2}$  being of the form in (50), which suggests that we need to regard the null space of  $M_{r_0+2q,p}$ , where

$$M_{r_0+2q,p} = \begin{bmatrix} Q_2^\top (A - \lambda_{j+1} I_n) & -Q_2^\top X_p \\ X_{r_0+2q}^\top & 0 \end{bmatrix}. \quad (51)$$

Suppose that the columns of

$$S_{r_0+2q,p} = \begin{bmatrix} S_{r_0+2q,p}^{(1)} \\ S_{r_0+2q,p}^{(2)} \end{bmatrix} \begin{matrix} n \\ p \end{matrix}$$

form an orthonormal basis of  $\mathcal{N}(M_{r_0+2q,p})$ . Then the assigning procedure is similar as that in Subsection III-A2, which is accomplished by distinguishing three different cases:  $\text{rank}(S_{r_0+2q,p}^{(1)}) \geq 2$ ,  $\text{rank}(S_{r_0+2q,p}^{(1)}) = 1$  and  $\text{Re}(u)$  and  $\text{Im}(u)$  are linearly independent with  $u$  being the left singular vector of  $S_{r_0+2q,p}^{(1)}$  corresponding to its only nonzero singular value, and otherwise.

Guaranteed by Theorem 2 below, we can proceed with the above assigning procedure till all columns of  $X$  and  $T$  corresponding to  $\{\lambda_{j+1}, \bar{\lambda}_{j+1}\}$  are acquired, which eventually yields  $T(r_0 + 1 : r_0 + 2a_{j+1}, r_0 + 1 : r_0 + 2a_{j+1})$  being of the special form specified in (47). And we recapitulate the assigning process of the repeated complex poles  $\{\lambda_{j+1}, \bar{\lambda}_{j+1}\}$  in Algorithm 4.

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**Algorithm 4** Assigning complex conjugate  $\{\lambda_{j+1}, \bar{\lambda}_{j+1}\}$ 


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**Input:**

$A, Q_2, X_{r_0}, T_{r_0}, \lambda_{j+1} \in \mathbb{C}$  with  $\text{Im}(\lambda_{j+1}) \neq 0$  and  $a_{j+1}$  (the multiplicity of  $\lambda_{j+1}$ ).

**Output:**

Orthogonal  $X_{r_0+2a_{j+1}}$  and upper quasi-triangular  $T_{r_0+2a_{j+1}}$ .

- 1: Set  $l = k = r_0$ .
- 2: **while**  $l < r_0 + 2a_{j+1}$  **do**
- 3: Find

$$S_{l,k} = \begin{bmatrix} S_{l,k}^{(1)} \\ S_{l,k}^{(2)} \end{bmatrix} \begin{matrix} n \\ k \end{matrix},$$

whose columns form an orthonormal basis of the null space of

$$M_{l,k} = \begin{bmatrix} Q_2^\top (A - \lambda_{j+1} I_n) & -Q_2^\top X_k \\ X_l^\top & 0 \end{bmatrix}; \text{ compute the SVD}$$

of  $S_{l,k}^{(1)} = U_{l,k} \Sigma_{l,k} V_{l,k}^*$ .

- 4: **if**  $\text{rank}(S_{l,k}^{(1)}) \geq 2$  **then**
- 5: Compute the  $(l+1)$ -th and  $(l+2)$ -th columns of  $X_{r_0+2a_{j+1}}$  and  $T_{r_0+2a_{j+1}}$  as in **Case iii** in Subsection III-A2; set  $l = l + 2$ ;
- 6: **else if**  $\text{rank}(S_{l,k}^{(1)}) = 1$  and  $\text{Re}(U_{l,k} e_1)$ ,  $\text{Im}(U_{l,k} e_1)$  are linearly independent **then**
- 7: Compute the  $(l+1)$ -th and  $(l+2)$ -th columns of  $X_{r_0+2a_{j+1}}$  and  $T_{r_0+2a_{j+1}}$  as in **Case iv** in Subsection III-A2; set  $l = l + 2$ ;
- 8: **else**
- 9: Find

$$S_{l,l} = \begin{bmatrix} S_{l,l}^{(1)} \\ S_{l,l}^{(2)} \end{bmatrix} \begin{matrix} n \\ l \end{matrix},$$

whose columns form an orthonormal basis of the null space of  $M_{l,l} = \begin{bmatrix} Q_2^\top (A - \lambda_{j+1} I_n) & -Q_2^\top X_l \\ X_l^\top & 0 \end{bmatrix}$ ; compute the  $(l+1)$ -th and  $(l+2)$ -th columns of  $X_{r_0+2a_{j+1}}$  and  $T_{r_0+2a_{j+1}}$  as in **Case v** in Subsection III-A2; set  $k = l$  and  $l = l + 2$ .

- 10: **end if**
  - 11: **end while**
- 

**C. Theoretical support**

While assigning repeated real poles, the assigning procedure described in Subsections III-A1 and III-B1 can be carried on only if the ranks of  $S_{q,q}^{(1)}$  in (12) and  $S_{r_0,r_0}^{(1)}$  in (43) are nonzero, which is guaranteed by the following theorem.

**Theorem 1.** Assume that  $(A, B)$  is controllable. Suppose that the poles  $\lambda_1, \dots, \lambda_j \in \mathfrak{L}$ , with multiplicities  $a_1, \dots, a_j$ , respectively, have been assigned. Let  $x_1, \dots, x_r$  be the corresponding columns of  $X$  obtained from the assigning process in former subsections, where  $r = \sum_{k=1}^j a_k$ . Assume that  $\lambda \in \mathbb{R}$  is distinct from  $\lambda_1, \dots, \lambda_j$ , and has been assigned  $q$  times with the corresponding columns  $x_{r+1}, \dots, x_{r+q}$  ( $r+q < n$ ) in  $X$  being obtained. Denote  $X_{r+q} = [x_1 \ \dots \ x_{r+q}]$  and

$$M_{r+q,r+q} = \begin{bmatrix} Q_2^\top (A - \lambda I_n) & -Q_2^\top X_{r+q} \\ X_{r+q}^\top & 0 \end{bmatrix}.$$

Let the columns of

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{matrix} n \\ r+q \end{matrix}$$

be an orthonormal basis of  $\mathcal{N}(M_{r+q,r+q})$ . Then  $\dim(\mathcal{R}(S)) = m$  and  $S_1 \neq 0$ .

*Proof:* The conclusion  $\dim(\mathcal{N}(M_{r+q,r+q})) = m$  is just that  $M_{r+q,r+q}$  is of full row rank. Assume that  $u \in \mathbb{R}^{n-m}$  and  $v \in \mathbb{R}^{r+q}$  satisfy  $[u^\top \ v^\top] M_{r+q,r+q} = 0$ , that is,

$$u^\top Q_2^\top (A - \lambda I_n) + v^\top X_{r+q}^\top = 0, \quad (52a)$$

$$u^\top Q_2^\top X_{r+q} = 0. \quad (52b)$$

Post-multiplying  $X_{r+q}$  on both sides of (52a) gives

$$u^\top Q_2^\top (A - \lambda I_n) X_{r+q} + v^\top = 0. \quad (53)$$

Substituting  $Q_2^\top A X_{r+q} = Q_2^\top X_{r+q} T_{r+q}$  into (53) leads to  $v = 0$  and  $u^\top Q_2^\top (A - \lambda I_n) = 0$  by (52b). Thus  $u = 0$  since  $(A, B)$  is controllable. So  $M_{r+q,r+q}$  is of full row rank, and hence  $\dim(\mathcal{N}(M_{r+q,r+q})) = m$ .

Now we are to prove  $S_1 \neq 0$ . It holds obviously if  $(r+q) < m$ . We now consider the case when  $(r+q) \geq m$ . Assume that  $S_1 = 0$ , then  $\text{rank}(S_2) = m$  and  $Q_2^\top X_{r+q} S_2 = 0$ . Hence there must exist a nonsingular matrix  $W \in \mathbb{R}^{m \times m}$  such that

$$X_{r+q} S_2 = BW. \quad (54)$$

Since  $Q_2^\top A X_{r+q} = Q_2^\top X_{r+q} T_{r+q}$  with  $T_{r+q}$  being the  $(r+q) \times (r+q)$  leading principal submatrix of  $T$ , so there must exist a matrix  $K \in \mathbb{R}^{m \times (r+q)}$  such that

$$A X_{r+q} = X_{r+q} T_{r+q} + BK. \quad (55)$$

Post-multiplying  $S_2$  on both sides of (55) and substituting (54) into it give  $ABW = X_{r+q} T_{r+q} S_2 + BK S_2$ . Noticing that  $W$  is nonsingular, so

$$AB = X_{r+q} T_{r+q} S_2 W^{-1} + X_{r+q} S_2 W^{-1} K S_2 W^{-1}.$$

Denote  $G_1 = T_{r+q} S_2 W^{-1} + S_2 W^{-1} K S_2 W^{-1}$ , then it can be simply verified by induction that  $A^k B = X_{r+q} G_k$  with  $G_k = T_{r+q} G_{k-1} + S_2 W^{-1} K G_{k-1}$ . And this eventually leads to

$$[B \ AB \ \dots \ A^{n-1} B] = X_{r+q} L$$

for some  $L \in \mathbb{R}^{(r+q) \times mn}$ , which implies that  $\text{rank}([B \ AB \ \dots \ A^{n-1} B]) \leq (r+q) < n$ , contradicting with the controllability of  $(A, B)$ . Hence  $S_1 \neq 0$ . ■

While assigning non-real repeated poles, continuing the assigning process is based on the facts that the matrix  $M_{j,j}$ , appearing in Step 15 in Algorithm 2, satisfies that  $\dim(\mathcal{N}(M_{j,j})) = m$  and there exists  $[z^\top \ w^\top]^\top \in \mathcal{N}(M_{j,j})$  with  $z \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^j$ , such that  $z \neq 0$  and  $\text{Re}(z)$  and  $\text{Im}(z)$  are linearly independent. This also applies to Step 9 in Algorithm 4. The following Theorem then ensures that these processes can be continued.

**Theorem 2.** Assume that  $(A, B)$  is controllable. Let  $\{\lambda_1, \dots, \lambda_j\} \subset \mathfrak{L}$  be a self-conjugate subset with  $a_1, \dots, a_j$  being the multiplicities of  $\lambda_1, \dots, \lambda_j$ , respectively, and let  $x_1, \dots, x_r$  be the associate columns of  $X$  obtained from the assigning process in previous subsections, where  $r = \sum_{k=1}^j a_k$ . Assume that  $\lambda = \alpha + i\beta \in \mathbb{C}$  ( $\beta \neq 0$ ) is some pole distinct from  $\lambda_1, \dots, \lambda_j$ , and  $x_{r+1}, x_{r+2}, \dots, x_{r+2q-1}, x_{r+2q}$  ( $r+2q < n$ ) are the columns of  $X$  corresponding to complex conjugate pairs  $\{\lambda, \bar{\lambda}\}$ . Define

$$M_{r+2q,r+2q} = \begin{bmatrix} Q_2^\top (A - \lambda I_n) & -Q_2^\top X_{r+2q} \\ X_{r+2q}^\top & 0 \end{bmatrix},$$

and let the columns of

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \begin{matrix} n \\ r+2q \end{matrix}$$

be an orthonormal basis of  $\mathcal{N}(M_{r+2q,r+2q})$ , then we have

- (1)  $\dim(\mathcal{R}(S)) = m$ ;
- (2)  $S_1 \neq 0$ ;
- (3) there exist  $0 \neq z = \text{Re}(z) + i\text{Im}(z) \in \mathbb{C}^n$  and  $w \in \mathbb{C}^{r+2q}$  with  $\text{Re}(z)$  and  $\text{Im}(z)$  being linearly independent, such that  $\begin{bmatrix} z^\top & w^\top \end{bmatrix}^\top \in \mathcal{R}(S)$ .

*Proof:* We can prove the (1), (2) results by the method proving Theorem 1, and we skip the proof process here.

Regarding (3), if  $\dim(\mathcal{N}(Q_2^\top X_{r+2q})) < (m-1)$ , then there exist two vectors  $\begin{bmatrix} z_1^\top & w_1^\top \end{bmatrix}^\top, \begin{bmatrix} z_2^\top & w_2^\top \end{bmatrix}^\top \in \mathcal{R}(S)$  with  $0 \neq z_1 \in \mathbb{C}^n$ ,  $0 \neq z_2 \in \mathbb{C}^n$ , and  $z_1, z_2$  being linearly independent. Let  $\begin{bmatrix} z^\top & w^\top \end{bmatrix}^\top = (\xi_1 + i\eta_1) \begin{bmatrix} z_1^\top & w_1^\top \end{bmatrix}^\top + (\xi_2 + i\eta_2) \begin{bmatrix} z_2^\top & w_2^\top \end{bmatrix}^\top$ ,  $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}$ , then we can always find suitable  $\xi_1, \xi_2, \eta_1, \eta_2$  such that the real part and the imaginary part of the resulted  $z$  are linearly independent. If  $\dim(\mathcal{N}(Q_2^\top X_{r+2q})) = (m-1)$ , assume that  $w_1, \dots, w_{m-1} \in \mathbb{C}^{r+2q}$  form an orthonormal basis of  $\mathcal{N}(Q_2^\top X_{r+2q})$  and  $0 \neq z = (1 + i\zeta)y$ ,  $y \in \mathbb{R}^n$ ,  $w \in \mathbb{C}^{r+2q}$  satisfy  $\begin{bmatrix} z^\top & w^\top \end{bmatrix}^\top \in \mathcal{R}(S)$  with  $\|z\|_2^2 + \|w\|_2^2 = 1$ . Obviously, it holds that  $Q_2^\top (A - \alpha I_n)y + \beta \zeta Q_2^\top y = Q_2^\top X_{r+2q} \text{Re}(w)$  and  $\zeta Q_2^\top (A - \alpha I_n)y - \beta Q_2^\top y = Q_2^\top X_{r+2q} \text{Im}(w)$ . Thus there exist  $u, v \in \mathbb{R}^m$  such that

$$\begin{cases} (A - \alpha I_n)y + \beta \zeta y - X_{r+2q} \text{Re}(w) &= Bu, \\ \zeta(A - \alpha I_n)y - \beta y - X_{r+2q} \text{Im}(w) &= Bv. \end{cases} \quad (56)$$

It follows from (56) that

$$\begin{aligned} & \beta(1 + \zeta^2)y + X_{r+2q}(\text{Im}(w) - \zeta \text{Re}(w)) \\ &= \zeta Bu - Bv, \end{aligned} \quad (57a)$$

$$\begin{aligned} & (1 + \zeta^2)(A - \alpha I_n)y - X_{r+2q}(\zeta \text{Im}(w) + \text{Re}(w)) \\ &= Bu + \zeta Bv. \end{aligned} \quad (57b)$$

Since  $Q_2^\top X_{r+2q} \begin{bmatrix} w_1 & \dots & w_{m-1} \end{bmatrix} = 0$ , hence

$$X_{r+2q} \begin{bmatrix} w_1 & \dots & w_{m-1} \end{bmatrix} = BG \quad (58)$$

for some  $G \in \mathbb{R}^{m \times (m-1)}$  with  $\text{rank}(G) = m-1$ . And it follows from  $Q_2^\top AX_{r+2q} = Q_2^\top X_{r+2q} T_{r+2q}$  that

$$AX_{r+2q} = X_{r+2q} T_{r+2q} + BZ \quad (59)$$

for some  $Z \in \mathbb{R}^{m \times (r+2q)}$ . Now define

$$Y = \left[ \begin{array}{c|c} \begin{matrix} w_1 & \dots & w_{m-1} \end{matrix} & \text{Im}(w) - \zeta \text{Re}(w) \\ \hline 0 & \beta(1 + \zeta^2) \end{array} \right],$$

$$\begin{aligned} L &= \begin{bmatrix} G & \zeta u - v \\ T_{r+2q} & \frac{1}{1+\zeta^2}(\zeta \text{Im}(w) + \text{Re}(w)) \end{bmatrix}, \\ M &= \begin{bmatrix} 0 & \alpha \end{bmatrix}, \\ E &= \begin{bmatrix} Z & \frac{1}{1+\zeta^2}(u + \zeta v) \end{bmatrix}. \end{aligned}$$

Noting (57a), (57b), (58) and (59), then the following equations

$$\begin{aligned} & \begin{bmatrix} X_{r+2q} & y \end{bmatrix} Y = BL, \\ & A \begin{bmatrix} X_{r+2q} & y \end{bmatrix} = \begin{bmatrix} X_{r+2q} & y \end{bmatrix} M + BE \end{aligned} \quad (60)$$

hold, where  $L$  is nonsingular since  $\begin{bmatrix} X_{r+2q} & y \end{bmatrix}$  is of full column rank. Then (60) shows that  $AB = \begin{bmatrix} X_{r+2q} & y \end{bmatrix} H_1$  with  $H_1 = MYL^{-1} + YL^{-1}EYL^{-1}$ . Hence by induction, we will get that  $A^{l+1}B = \begin{bmatrix} X_{r+2q} & y \end{bmatrix} H_{l+1}$ , where  $H_{l+1} = MH_l + YL^{-1}EH_l$  with  $l \geq 1$ . Eventually,  $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} X_{r+2q} & y \end{bmatrix} \begin{bmatrix} YL^{-1} & H_1 & \dots & H_{n-1} \end{bmatrix}$ , suggesting that

$$\text{rank}(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}) < n.$$

This contradicts with the assumption that  $(A, B)$  is controllable. Thus we have proved (3). ■

## D. Algorithm

The framework of our algorithm referred to as ‘‘Schur-multi’’ is given in this subsection. We assume that repeated real poles appear together in  $\mathfrak{L}$ , while repeated complex conjugate poles appear in pairs, that is, they appear as  $\underbrace{\{\lambda, \bar{\lambda}\}, \dots, \{\lambda, \bar{\lambda}\}}_a$  in  $\mathfrak{L}$  adjacently, where  $a$  is

the counting time (the algebraic multiplicity) of  $\lambda$  (and  $\bar{\lambda}$ ) in  $\mathfrak{L}$ . The Schur-multi algorithm below combines techniques designed for simple poles in [11] and techniques for repeated poles in this paper. Again, we denote the multiplicity of  $\lambda_j \in \mathfrak{L}$  by  $a_j$ .

---

### Algorithm 5 Framework of our Schur-multi algorithm.

---

#### Input:

$A, B$  and  $\mathfrak{L} = \{\lambda_1, \dots, \lambda_n\}$ .

#### Output:

The feedback matrix  $F$ .

- 1: Compute the QR decomposition of  $B = Q \begin{bmatrix} R^\top & 0 \end{bmatrix}^\top = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R^\top & 0 \end{bmatrix}^\top = Q_1 R$ .
  - 2: **if**  $a_1 = 1$  **then**
  - 3:   Compute the initial columns of  $X$  and  $T$  by Schur-rob [11]; set  $j = 1$  for  $\lambda_1 \in \mathbb{R}$  and  $j = 2$  for  $\lambda_1 \in \mathbb{C}$ .
  - 4: **else if**  $\lambda_1 \in \mathbb{R}$  **then**
  - 5:   Compute  $X_{a_1}$  and  $T_{a_1}$  by Algorithm 1; set  $j = a_1$ .
  - 6: **else**
  - 7:   Compute  $X_{2a_1}$  and  $T_{2a_1}$  by Algorithm 2; set  $j = 2a_1$ .
  - 8: **end if**
  - 9: **while**  $j < n$  **do**
  - 10:   **if**  $a_{j+1} = 1$  **then**
  - 11:     Compute the corresponding columns of  $X$  and  $T$  by Schur-rob [11]; set  $j = j+1$  for  $\lambda_{j+1} \in \mathbb{R}$  and  $j = j+2$  for  $\lambda_{j+1} \in \mathbb{C}$ .
  - 12:   **else if**  $\lambda_{j+1} \in \mathbb{R}$  **then**
  - 13:     Compute  $X_{j+a_{j+1}}$  and  $T_{j+a_{j+1}}$  by Algorithm 3; set  $j = j + a_{j+1}$ .
  - 14:   **else**
  - 15:     Compute  $X_{j+2a_{j+1}}$  and  $T_{j+2a_{j+1}}$  by Algorithm 4; set  $j = j + 2a_{j+1}$ .
  - 16:   **end if**
  - 17: **end while**
  - 18: Compute  $F$  by  $F = R^{-1}Q_1^\top (X_n T_n X_n^\top - A)$ .
- 

## IV. NUMERICAL EXAMPLES

In this section, we illustrate the performance of our Schur-multi method by comparing with the MATLAB functions place [14], robpole [28] and the Schur-rob method [11] on some examples.

Similarly to [11], we define

$$\text{prec}s = \left\lceil \max_{1 \leq j \leq n} \left( \log \left( \left| \frac{\lambda_j - \hat{\lambda}_j}{\lambda_j} \right| \right) \right) \right\rceil$$

to characterize the precision of the assigned poles, where  $\hat{\lambda}_j$ ,  $j = 1, \dots, n$ , are the computed eigenvalues of the obtained closed-loop system matrix  $A_c = A + BF$ . Actually,  $\text{prec}s$  is the ceiling value of the exponent of the maximum relative error of  $\hat{\lambda}_j$  ( $j = 1, \dots, n$ ), relative to the entries in  $\mathfrak{L}$ . Obviously, smaller  $\text{prec}s$  would imply more accurately computed poles. Regarding the robustness of the closed-loop system, different measures are used in these methods for solving the SFRPA. We will compare three measures for all methods. Specifically, assume that the spectral decomposition and

the real Schur decomposition of  $A_c = A + BF$  respectively are

$$A + BF = X\Lambda X^{-1}, \quad A + BF = UTU^\top,$$

where  $\Lambda$  is diagonal,  $T$  is upper quasi-triangular and  $U$  is orthogonal. Then the measures adopted in `place` and `robpole` are closely related to the condition number of the eigenvectors matrix  $X$ , i.e.  $\kappa_F(X) = \|X\|_F \|X^{-1}\|_F$ , while `Schur-rob` and our `Schur-multi` aim to minimize the departure from normality of  $A_c$  (denoted by “*dep.*”). We also display the Frobenius norm of the feedback matrix  $F$  (denoted by “ $\|F\|_F$ ”), which is also regarded as a measure of robustness in some literature. In addition, the CPU time for all methods is also presented. When `robpole` is applied, the maximum number of sweep is set to be the default value 5 for all examples. All calculations are carried out by running MATLAB R2012a, with machine epsilon  $\epsilon \approx 2.2 \times 10^{-16}$ , on an Intel®Core™i3, dual core, 2.27 GHz machine, with 2.00 GB RAM.

The first illustrative set includes CARE examples 1.6, 2.9 #1[1] and DARE example 1.12 [2], in which some poles are repeated and real. Additional, in the following TABLE I and TABLE II, we will use  $\alpha(k)$  to represent  $\alpha \times 10^k$  for space saving.

**Example IV.1.** The three examples in this test set come from the SLICOT CARE/DARE benchmark collections [1], [2]. The numerical results on precision and robustness for these four algorithms are exhibited in TABLE I. Concerning the CARE example 2.9 #1, compared with `Schur-rob`, our `Schur-multi` does not make improvement on “*prec*”. The reason might be that some poles are rather close to the imaginary axis. This is a weakness of the Schur-type methods. Note that we do not list the “*prec*” values for the DARE example 1.12 since some algorithms could not achieve any relative accuracy for certain assigned poles. And in TABLE II, we display the differences between the placed poles and the eigenvalues of the computed  $A_c$  obtained from distinct methods. The “exact poles” column gives the exact values of the poles to be assigned. TABLE II shows that our `Schur-multi` produces the best result on this example.

All test sets in the following two examples are randomly generated by the “`randn`” command in MATLAB, where  $\mathcal{L}$  contains some repeated poles (real or non-real).

**Example IV.2.** This example consists of two test sets. The first test set, which is to illustrate the performance of all methods when repeated poles are all real, contains 70 random examples, where  $n$  varies from 3 to 13 increased by 2, and  $m$  is set to be 2,  $\lfloor \frac{n}{2} \rfloor, n-1$  for each  $n$ . For each fixed  $(n, m)$ , the greatest multiplicity  $a_{max}$  of all real poles increases from 1 to  $m$  in increment of 1. All examples are generated as follows. We first randomly generate a nonsingular matrix  $Y \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}, F \in \mathbb{R}^{m \times n}$  by the MATLAB function `randn` and the assigned poles  $\mathcal{L} = \{\text{randn} \times \text{ones}(1, a_{max}), \text{randn}(1, n - a_{max})\}$ , then set  $A = Y\Lambda Y^{-1} - BF$ , where the diagonal elements of the diagonal matrix  $\Lambda$  are those in  $\mathcal{L}$ . Taking  $A, B$  and  $\mathcal{L}$  as the input, we apply the methods `place`, `robpole`, `Schur-rob` and `Schur-multi` to these examples, where the poles are assigned in ascendant order.

For concision, we only list results for  $n = 13$ . Results for other examples are quite similar. Specifically, Fig. 1 to Fig. 4 show the three measures of robustness and the precision of the computed poles by all four methods, and Fig. 5 plots the ratios of the CPU time costs of `place`, `robpole` and `Schur-rob` with respect to that of `Schur-multi`. In each figure, the three subfigures correspond to  $m = 2, 6$  and 12, respectively. The  $x$ -axis represents  $a_{max}$ , and the values in the  $y$ -axis are mean values over 50 trials for a certain triple  $(13, m, a_{max})$ .

On these examples, our method is comparable with `place` and `robpole`, but with much less time cost. Comparing with `Schur-rob`, `Schur-multi` does improve the relative accuracy of the assigned poles when some poles to be assigned are repeated and real.

The second test set consists of 82 random examples, which is to demonstrate the performance of all methods when non-real repeated poles are contained in  $\mathcal{L}$ . Here, we take  $n$  varying from 7 to 19 with an increment of 2, and  $m$  is set to be 3,  $\lfloor \frac{n}{2} \rfloor, n-1$  for each  $n$ . For fixed  $(n, m)$ , the largest multiplicity  $a_{max}$  of all complex poles increases from 2 to  $\min\{\lfloor \frac{n}{2} \rfloor, m\}$ . All examples are generated as follows. First, we randomly generate the placed poles  $\mathcal{L} = \{\text{randn}(1, n - 2a_{max}), \lambda \times \text{ones}(1, a_{max}), \bar{\lambda} \times \text{ones}(1, a_{max})\}$  with  $\lambda = \text{randn} + i \times \text{randn}$ , and three matrices  $Y \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, F \in \mathbb{R}^{m \times n}$  using the MATLAB function `randn`. Compute the QR decomposition of  $Y$  as  $Y = Q_Y R_Y$ , and we reset the diagonal and subdiagonal entries of  $R_Y$  such that it is upper quasi-triangular with its eigenvalues being those in  $\mathcal{L}$ . Then set  $A = Q_Y R_Y Q_Y^\top - BF$ . Thereafter, the algorithms `place`, `robpole`, `Schur-rob` and `Schur-multi` are applied on all examples with  $A, B$  and  $\mathcal{L}$  taken as the input.

Fig. 6 to Fig. 10 exhibit the numerical results on *dep.*,  $\|F\|_F$ , and  $\kappa_F(X)$ , *prec*s and the CPU time ratio for  $n = 19$ , respectively, where the  $x$ -axis and the  $y$ -axis own the some meanings as those in the first test set. Each figure includes three subfigures, where the first one displays the results for  $m = 3$ , the second for  $m = 9$  and the third for  $m = 18$ . Note that for the CPU time, we still adopt the time cost of `Schur-multi` as the standard of comparison, and present the ratios of `place`, `robpole` and `Schur-rob` to it.

All figures show that when  $a_{max}$  is no more than  $\lfloor \frac{m+1}{2} \rfloor$ , then compared with `robpole`, our approach produces comparable results on the robustness and the precision of the assigned poles, but with much less time consumption. However, if there exists at least one complex pole with its multiplicity being larger than  $\lfloor \frac{m+1}{2} \rfloor$ , the closed-loop system matrix obtained by `Schur-multi` can not be diagonalized and it would not be as robust as that computed by `robpole`. Notice that for our `Schur-multi` method, there are sharp jumps in Fig. 6 and Fig. 7 for  $m = 9, 18$  cases, where  $a_{max} = \lfloor \frac{m+1}{2} \rfloor$ . And the explanation for those jumps is:  $\lfloor \frac{m+1}{2} \rfloor$  actually is a threshold that distinguishes if the repeated non-real pole acts as a semi-simple eigenvalue or not, hence those repeated complex poles, whose multiplicities equal to  $\lfloor \frac{m+1}{2} \rfloor$ , would be more sensitive to perturbations; and such behavior eventually reflects in *dep.* and  $\|F\|_F$ . In addition, compared with `Schur-rob`, `Schur-multi` does make some improvements on the precision of the assigned repeated complex conjugate poles. The undisplayed results for other different  $n$  show similar behavior.

It is well known that `place` and `robpole` can not solve the **SFRPA** if the multiplicity of some pole is greater than  $m$ , while `Schur-rob` and our `Schur-multi` can still work. The following randomly generated examples are to reveal the behavior of `Schur-rob` and `Schur-multi` on examples in which the multiplicity of some repeated pole might be greater than  $m$ .

**Example IV.3.** This example also consists of two test sets. The first test set, where the repeated poles are all real, is comprised of 270 random examples with  $n$  increasing from 7 to 27 in increment of 4, and  $m$  being 2,  $\lfloor \frac{n}{2} \rfloor, n-1$  for each  $n$ . For fixed  $(n, m)$ , the greatest multiplicity of the assigned repeated real poles  $a_{max}$  varies from 2 to  $n-1$ . All examples are generated as below. We first randomly generate the assigned poles  $\mathcal{L} = \{\text{randn} \times \text{ones}(1, a_{max}), \text{randn}(1, n - a_{max})\}$  and  $Y \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, F \in \mathbb{R}^{m \times n}$  by the MATLAB



TABLE I: Numerical results for four algorithms on CARE/DARE examples

	CARE example 1.6				CARE example 2.9 #1				DARE example 1.12		
	<i>prec</i> s	<i>dep.</i>	$\kappa_F(X)$	$\ F\ _F$	<i>prec</i> s	<i>dep.</i>	$\kappa_F(X)$	$\ F\ _F$	<i>dep.</i>	$\kappa_F(X)$	$\ F\ _F$
place	-11	1.5(6)	1.7(15)	2.2(3)	-11	2.9(6)	8.5(4)	2.8(1)	4.3(7)	9.2(292)	4.3(7)
robpole	-13	7.5(5)	2.2(7)	2.2(2)	-12	2.9(6)	8.9(4)	2.8(1)	3.9(12)	1.3(308)	3.9(12)
Schur-rob	-8	1.1(5)	9.0(7)	1.2(2)	-9	7.3(6)	2.0(6)	2.9(1)	9.8(0)	5.6(292)	6.5(0)
Schur-multi	-11	2.6(5)	1.3(7)	4.5(2)	-9	2.6(6)	1.2(6)	2.8(1)	9.1(0)	3.2(295)	5.5(0)

TABLE II: Accuracy of the assigned poles for DARE example 1.12

<i>num.</i>	exact poles	$\lambda_j - \hat{\lambda}_j$			
		place	robpole	Schur-rob	Schur-multi
1	8.1(-1)	-3.3(-16)	-3.3(-16)	-3.3(-16)	-3.3(-16)
2	5.8(-1)	-2.5(-7)	3.6(-5)	-1.4(-12)	2.3(-13)
3	1.1(-3)	8.4(-4)	2.9(-4)	-1.5(-4)	-6.4(-5)
4	0	-3.4(-17)	-3.4(-17)	-3.4(-17)	-3.4(-17)
5	0	-5.2(-17)	-5.2(-17)	-5.2(-17)	-5.2(-17)
6	7.6(-1)+i×1.4(-1)	1.9(-7)+i×1.2(-7)	-4.6(-5)+i×3.7(-6)	-7.1(-13)+i×1.3(-13)	6.2(-13)+i×4.8(-13)
7	7.6(-1)+i×1.4(-1)	1.9(-7)+i×1.2(-7)	-4.6(-5)+i×3.7(-6)	-7.1(-13)+i×1.3(-13)	6.2(-13)+i×4.8(-13)
8	6.4(-1)+i×2.3(-1)	-2.5(-8)+i×3.1(-8)	-4.0(-5)+i×1.6(-5)	9.3(-13)+i×1.1(-12)	-6.4(-13)+i×3.5(-13)
9	6.4(-1)+i×2.3(-1)	-2.5(-8)+i×3.1(-8)	-4.0(-5)+i×1.6(-5)	9.3(-13)+i×1.1(-12)	-6.4(-13)+i×3.5(-13)
10	-9.0(-4)+i×6.6(-4)	-8.3(-4)+i×6.6(-4)	-9.0(-4)+i×6.6(-4)	1.2(-4)+i×8.8(-5)	5.2(-5)+i×3.9(-5)
11	-9.0(-4)+i×6.6(-4)	2.0(-3)+i×6.6(-4)	4.0(-4)+i×6.6(-4)	1.2(-4)+i×8.8(-5)	5.2(-5)+i×3.9(-5)
12	3.5(-4)+i×1.1(-3)	7.1(-4)+i×1.2(-4)	-8.1(-1)+i×1.1(-3)	-4.7(-5)+i×1.4(-4)	-2.1(-5)+i×6.1(-5)
13	3.5(-4)+i×1.1(-3)	7.1(-4)+i×1.2(-4)	3.5(-4)+i×1.1(-3)	-4.7(-5)+i×1.4(-4)	-2.1(-5)+i×6.1(-5)

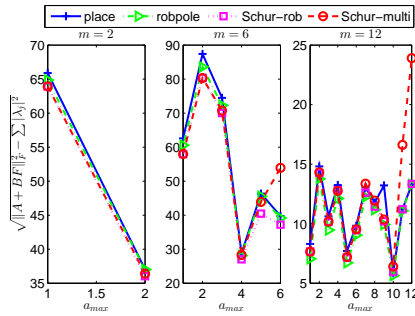
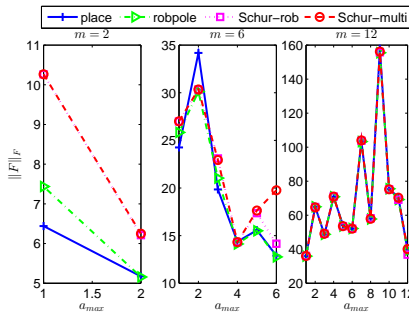
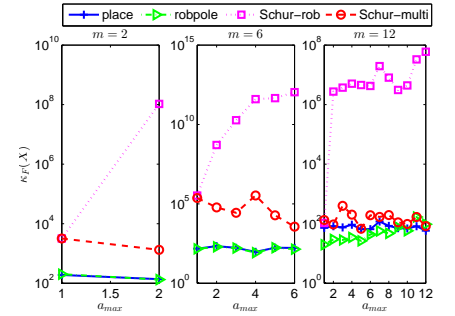
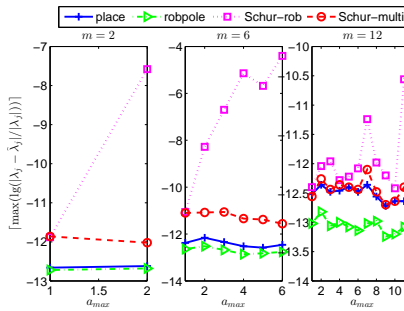
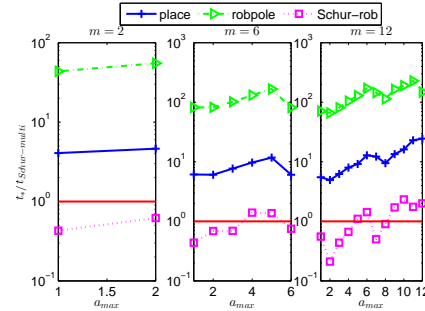
Fig. 1: *dep.* (Example IV.2 with real repeated poles)Fig. 2:  $\|F\|_F$  (Example IV.2 with real repeated poles)Fig. 3:  $\kappa_F(X)$  (Example IV.2 with real repeated poles)Fig. 4: *prec*s (Example IV.2 with real repeated poles)

Fig. 5: CPU time ratio (Example IV.2 with real repeated poles)

function `randn`. Then we compute the QR decomposition of  $Y$  as  $Y = Q_Y R_Y$ , reset the diagonal elements of the upper triangular matrix  $R_Y$  be those in  $\mathcal{L}$ , and set  $A = Q_Y R_Y Q_Y^\top - BF$ . Taking  $A, B$  and  $\mathcal{L}$  as the input, we then apply `Schur-rob` and `Schur-multi` to all generated examples. The poles in  $\mathcal{L}$  are also assigned in ascendant order. Note that when applying `place` and `robpole` on these examples, they fail to give results for some examples. For

instance, when  $m = 2$  and  $a_{max} > 2 = m$ , they fail to output solutions.

Both algorithms produce fairly similar *dep.* and  $\|F\|_F$ , and we omit the interrelated results here. The numerical results on  $\kappa_F(X)$  and *prec*s with respect to  $a_{max}$  for  $n = 19$  are displayed in Fig. 11 and Fig. 12, respectively, where the  $x$ -axis and  $y$ -axis own the same meanings as those in Example IV.2. In each figure, the three

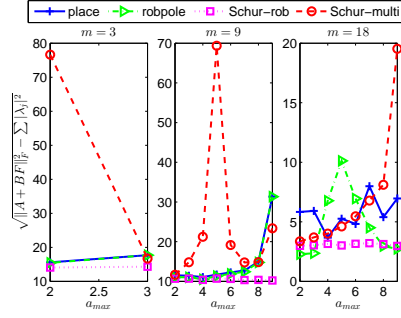


Fig. 6: *dep.* (Example IV.2 with non-real repeated poles)

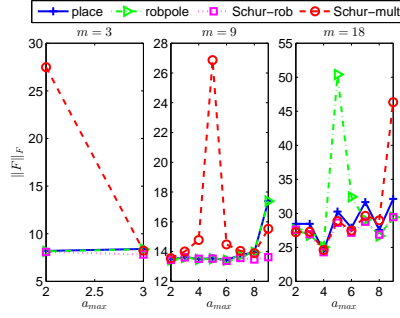


Fig. 7:  $\|F\|_F$  (Example IV.2 with non-real repeated poles)

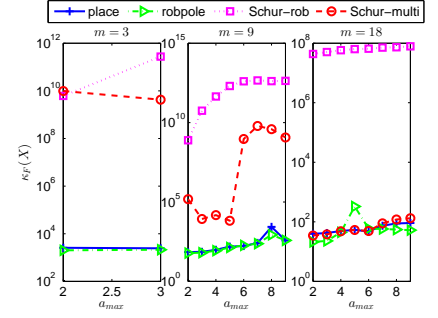


Fig. 8:  $\kappa_F(X)$  (Example IV.2 with non-real repeated poles)

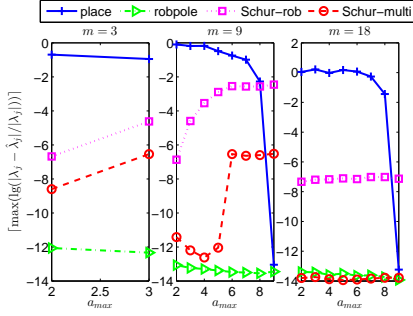


Fig. 9: *prec\_s* (Example IV.2 with non-real repeated poles)

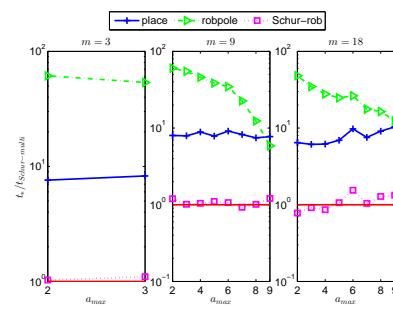


Fig. 10: CPU time ratio (Example IV.2 with non-real repeated poles)

subfigures correspond to  $m = 2, 9$  and  $18$ , respectively.

From Fig. 11 and Fig. 12, we know that the condition numbers of the eigenvectors matrices obtained by Schur-multi are smaller than those by Schur-rob, and the eigenvalues of  $A_c$  computed by Schur-multi are more accurate than those by Schur-rob. The differences become more significant when  $a_{max}$  is no greater than  $m$ . If  $a_{max}$  is greater than  $m$ , that is, some eigenvalues of  $A_c$  are defective, the precision of the poles diminishes. For other  $(n, m, a_{max})$ ,  $\kappa_F(X)$  and *prec\_s* show quite similar variation tendency.

It is shown in Subsection III-A1 that if the repeated real pole with multiplicity  $a_{max}$  is assigned as the initial  $\lambda_1$ , then its geometric multiplicity is theoretically  $\min\{m, a_{max}\}$ . However, if it is not assigned foremost, we cannot prove such result in theory. We then compute the geometric multiplicity (denoted as " $g_{multi}$ ") of the repeated real pole by using the SVD of  $(A_c - \lambda I_n)$ , where  $A_c$  is the computed closed-loop system matrix and  $\lambda \in \mathcal{L}$ . Note that in our experiments, the poles are assigned in ascendant order. That is, the repeated real pole may not be the first one to be placed. However, the numerical results for  $n = 19$  listed in TABLE III show that  $g_{multi}$  obtained by Schur-multi always equals to  $\min\{m, a_{max}\}$ . The unshown results for other different  $(n, m, a_{max})$  behave similarly.

All numerical examples in the second test set are designed to illustrate the behavior of both Schur-type approaches when  $\mathcal{L}$  contains some repeated complex conjugate poles with their multiplicities exceeding  $m$ . There are 193 random illustrative examples in this test set, with  $n$  increasing from 7 to 25 in an increment of 2, and  $m$  taking 3,  $\lfloor \frac{n}{2} \rfloor$ ,  $n-1$  for each  $n$ . With  $(n, m)$  fixed, the largest multiplicity of the assigned complex poles varies from 2 to  $\lfloor \frac{n}{2} \rfloor$ . All these examples are generated in the same way as those in the second test set in Example IV.2. Regarding  $A, B$  and  $\mathcal{L}$  as the input, Schur-rob and Schur-multi are then applied to each example.

Here, we just exhibit the numerical results for  $n = 25$ . Numerical

results on *dep.*,  $\|F\|_F$  and  $\kappa_F(X)$  for both algorithms are shown in Fig. 13 to Fig. 15, and Fig. 16 displays the relative accuracy *prec\_s* of the assigned poles. Each figure includes three subfigures, corresponding to  $m = 3, 12$  and  $24$ , respectively. The  $x$ -axis and  $y$ -axis own the same meanings as those in Example IV.2. From these figures we can see that Schur-multi produces slightly worse, but comparable *dep.* and  $\|F\|_F$  as Schur-rob, while  $\kappa_F(X)$  and *prec\_s* produced by Schur-multi are much better than those by Schur-rob. Numerical results for other  $n$  behave similarly.

When the largest multiplicity of the repeated non-real poles is larger than  $\lfloor \frac{m+1}{2} \rfloor$ , for the computed  $A_c$  by Schur-multi, there exist defective complex conjugate eigenvalues. Consequently, the relative accuracy of the placed repeated complex conjugate poles would be not that high. To show the geometric multiplicity (denoted as " $g_{multi}$ ") of non-real repeated eigenvalues of  $A_c$  visually, just as what we do in the first test set, we shall compute it by using the SVD of  $(A_c - \lambda I_n)$ , where  $A_c$  is the computed closed-loop system matrix and  $\lambda \in \mathcal{L}$  with  $\text{Im}(\lambda) \neq 0$ . Typically, relevant results for  $n = 25$  are displayed in TABLE IV, which shows that  $g_{multi}$  obtained from Schur-multi equals to the smaller value between its corresponding algebraic multiplicity and  $\lfloor \frac{m+1}{2} \rfloor$ . The unshown results for other different  $(n, m, a_{max})$  are quite similar.

## V. CONCLUSION

Based on the Schur-rob method [11], a refined approach is proposed to solve the **SFRPA**, specifically when some poles to be assigned are repeated. In the proposed Schur-multi method, we treat the geometric multiplicities of the repeated poles as the precedential consideration, and then try to minimize the departure from normality of the closed-loop system matrix  $A_c$ . Numerical results show that the Schur-multi method does outperform the

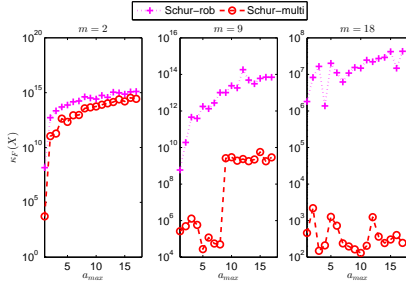


Fig. 11:  $\kappa_F(X)$  (Example IV.3 with real repeated poles)

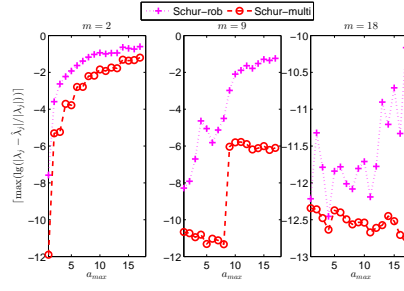


Fig. 12:  $prec_s$  (Example IV.3 with real repeated poles)

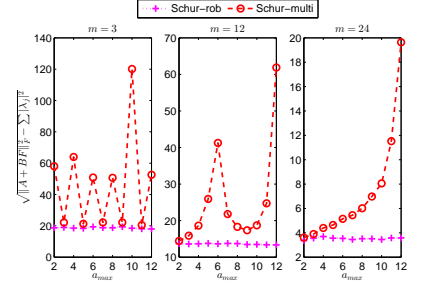


Fig. 13:  $dep.$  (Example IV.3 with non-real repeated poles)

TABLE III: Geometric multiplicity over 50 trials (real repeated poles)

$a_{max}$	$g_{multi}$ for $n = 19$					
	$m = 2$		$m = \lfloor \frac{n}{2} \rfloor$		$m = n - 1$	
	Schur-rob	Schur-multi	Schur-rob	Schur-multi	Schur-rob	Schur-multi
2	1.04	2.00	1.44	2.00	1.96	2.00
3	1.06	2.00	2.10	3.00	2.80	3.00
4	1.04	2.00	2.44	4.00	3.86	4.00
5	1.06	2.00	2.22	5.00	4.98	5.00
6	1.06	2.00	2.90	6.00	5.90	6.00
7	1.08	2.00	4.24	7.00	6.88	7.00
8	1.06	2.00	4.28	8.00	7.92	8.00
9	1.08	2.00	4.42	9.00	8.92	9.00
10	1.02	2.00	5.06	9.00	9.86	10.00
11	1.16	2.00	4.98	9.00	10.84	11.00
12	1.10	2.00	5.54	9.00	11.90	12.00
13	1.14	2.00	5.60	9.00	12.84	13.00
14	1.14	2.00	6.66	9.00	13.70	14.00
15	1.20	2.00	6.62	9.00	14.76	15.00
16	1.28	2.00	7.78	9.00	15.66	16.00
17	1.30	2.00	8.20	9.00	16.66	17.00
18	1.44	2.00	8.46	9.00	17.32	18.00

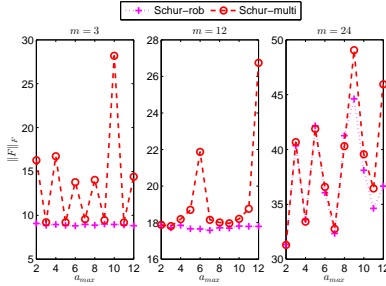


Fig. 14:  $\|F\|_F$  (Example IV.3 with non-real repeated poles)

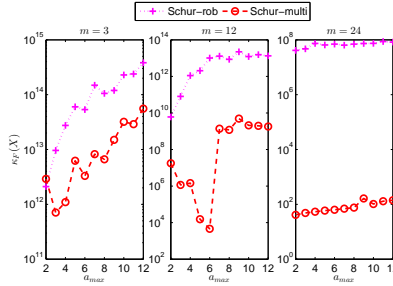


Fig. 15:  $\kappa_F(X)$  (Example IV.3 with non-real repeated poles)

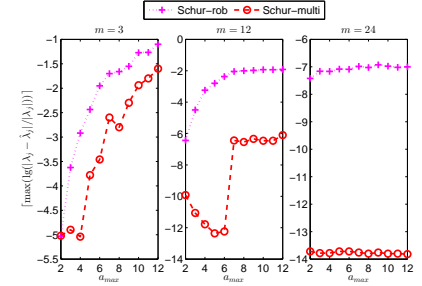


Fig. 16:  $prec_s$  (Example IV.3 with non-real repeated poles)

Schur-rob method for examples with repeated poles. Moreover, our Schur-multi method can still produce fairly good results when place and robpole fail for examples where the multiplicity of the repeated pole is greater than  $m$ .

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TABLE IV: Geometric multiplicity over 50 trials (non-real repeated poles)

$a_{max}$	$g_{multi}$ for $n = 25$					
	$m = 3$		$m = \lfloor \frac{n}{2} \rfloor$		$m = n - 1$	
	Schur-rob	Schur-multi	Schur-rob	Schur-multi	Schur-rob	Schur-multi
2	1.00	2.00	1.00	2.00	1.00	2.00
3	1.00	2.00	1.00	3.00	2.00	3.00
4	1.00	2.00	1.00	4.00	3.00	4.00
5	1.00	2.00	1.00	5.00	4.00	5.00
6	1.00	2.00	1.00	6.00	5.00	6.00
7	1.00	2.00	1.00	6.00	6.00	7.00
8	1.00	2.00	1.06	6.00	7.00	8.00
9	1.00	2.00	2.04	6.00	8.00	9.00
10	1.00	2.00	3.08	6.00	9.00	10.00
11	1.00	2.00	4.04	6.00	10.00	11.00
12	1.00	2.00	5.16	6.00	11.00	12.00

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